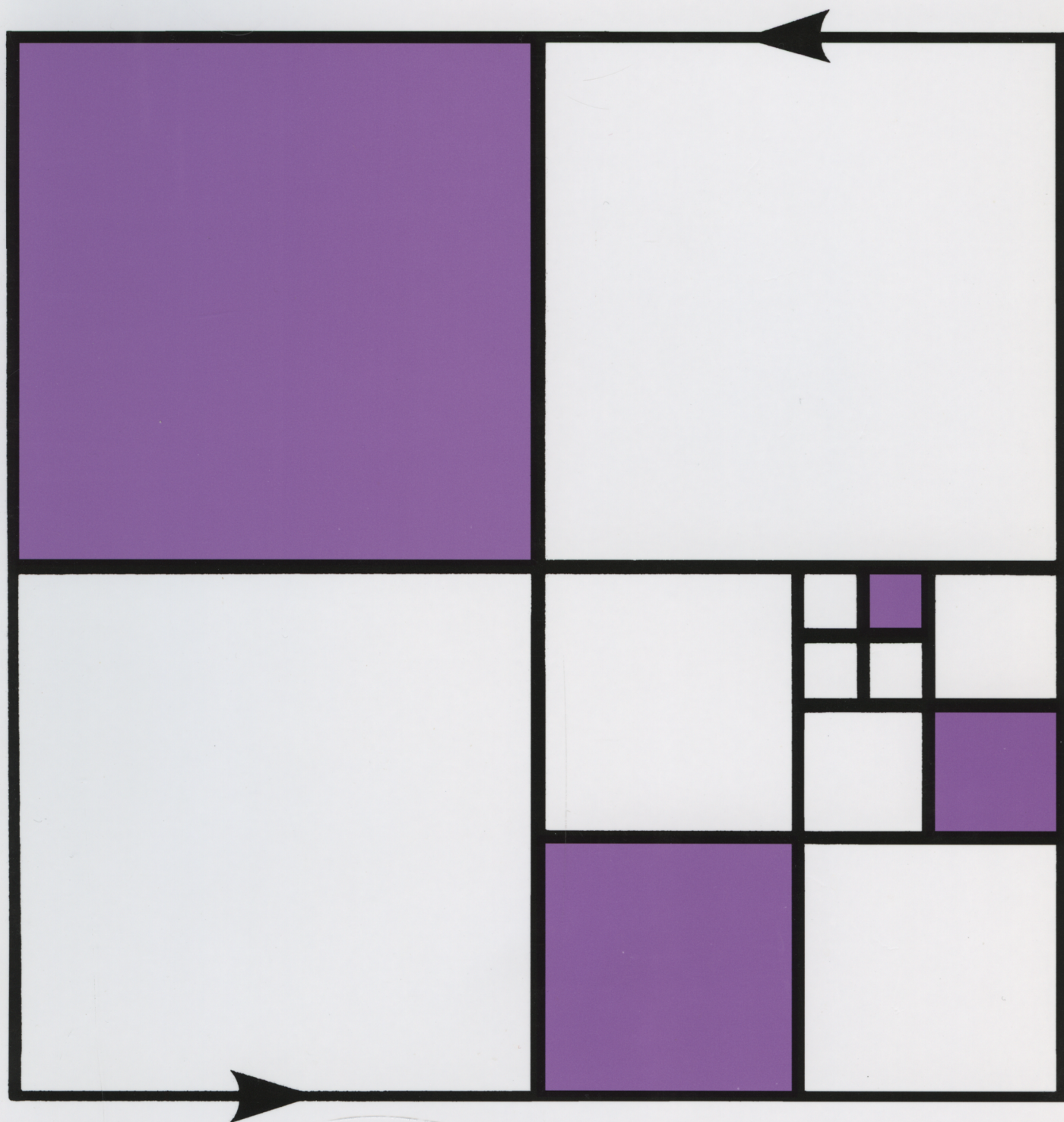


COMPLEX ANALYSIS

UNIT B1 INTEGRATION



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UNIT B1 INTEGRATION

Prepared by the Course Team

The Open University, Walton Hall, Milton Keynes, MK7 8AA.
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COMPLEX ANALYSIS COMPLEX ANALYSIS COMPLEX ANALYSIS

Before working through this text, make sure that you have read the
Course Guide for M337 Complex Analysis.

The Open University, Walton Hall, Milton Keynes, MK7 6AA.

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CONTENTS

Introduction	4
Study guide	5
1 Integrating Real Functions (audio-tape)	6
2 Integrating Complex Functions	10
2.1 Integration along a smooth path	10
2.2 Integration along a contour	15
2.3 Reverse paths and contours	18
3 Evaluating Contour Integrals	20
3.1 The Fundamental Theorem of Calculus	20
3.2 Closed paths and grid paths	24
3.3 Proof of the Grid Path Theorem	26
4 Estimating Contour Integrals	28
4.1 The length of a contour	28
4.2 The Estimation Theorem	29
4.3 Proof of the Estimation Theorem	32
Exercises	34
Solutions to the Problems	36
Solutions to the Exercises	40

INTRODUCTION

In this unit we introduce *complex integration*, an important concept which gives complex analysis its special flavour. We shall spend most of this unit setting up the complex integral, deriving its main properties, and illustrating various techniques for evaluating it. We discuss its applications in later units.

As in Block A, our initial aim is to try to extend ideas from real analysis to the arena of complex numbers and complex functions. Consider, for example, the fact that

$$\int_a^b x^2 dx = \frac{1}{3} (b^3 - a^3),$$

where a and b are real numbers (see Figure 0.1). We can express this by saying that

the integral of the function $f(x) = x^2$ over the interval $[a, b]$ is $\frac{1}{3}(b^3 - a^3)$.

In complex analysis, an analogous statement is that, if α and β are complex numbers, then

the integral of the function $f(z) = z^2$ along the line segment from α to β is $\frac{1}{3}(\beta^3 - \alpha^3)$.

We write this as

$$\int_{\Gamma} z^2 dz = \frac{1}{3} (\beta^3 - \alpha^3),$$

where the path Γ is the line segment from α to β (see Figure 0.2).

But this apparently straightforward idea leaves more questions unanswered than answered. For example, in the above real integral, the expression $\frac{1}{3}(b^3 - a^3)$ represents the area between the graph and the x -axis from $x = a$ to $x = b$. Is there a corresponding geometrical interpretation of the expression $\frac{1}{3}(\beta^3 - \alpha^3)$? It turns out, unfortunately, that there is no simple geometric interpretation of a complex integral, and it does not help us to understand complex integration if we try to think of such an interpretation.

Undeterred, let us continue ...

The path we considered above was the *line segment* from α to β . But there are many other paths from α to β . Do we get the same answer if we integrate the function $f(z) = z^2$ along other paths from α to β ? Indeed, what exactly does it mean to 'integrate a function along a path'?

These questions will be answered in this unit. Since the definition of a complex integral is based on that of a real integral, we devote Section 1 to a revision of real integration. In particular, we remind you of the definition of the *Riemann integral* and of some of its most important properties.

In Section 2, we introduce the concept of *complex integration*, and show you how to integrate a given complex function along a smooth path. We also extend these ideas to integration along a *contour* — a finite sequence of smooth paths laid end to end.

In Section 3, we prove the *Fundamental Theorem of Calculus*, which shows that integration and differentiation are essentially inverse processes. From this result, it follows that the integral of $f(z) = z^2$ along *any* contour from α to β is $\frac{1}{3}(\beta^3 - \alpha^3)$.

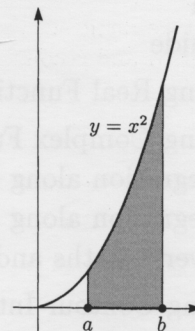


Figure 0.1

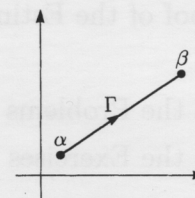


Figure 0.2

We shall need to be very careful about how we apply results such as the Fundamental Theorem of Calculus. For example, it follows from the above remarks that if we integrate the function $f(z) = z^2$ along any smooth path Γ whose endpoints α and β coincide (see Figure 0.3), then $\frac{1}{3}(\beta^3 - \alpha^3) = 0$, and so

$$\int_{\Gamma} z^2 dz = 0.$$

However, if we integrate the function $f(z) = 1/z$ along the smooth paths Γ_1 and Γ_2 shown in Figure 0.4, then

$$\int_{\Gamma_1} \frac{1}{z} dz = 0, \quad \text{but} \quad \int_{\Gamma_2} \frac{1}{z} dz = 2\pi i.$$

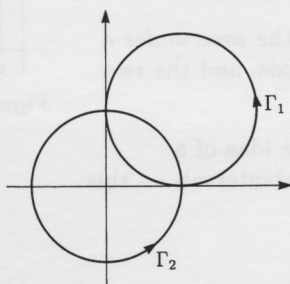


Figure 0.4

Thus, what works for the function $f(z) = z^2$ will not necessarily work for other functions.

Later in the course we shall meet many complex integrals whose values we cannot determine exactly. We can, however, *estimate* them, in the sense that we can derive an upper estimate for

$$\left| \int_{\Gamma} f(z) dz \right|,$$

the *modulus* of the integral. In Section 4 we present the *Estimation Theorem*, which gives such an upper estimate, and which will be an important tool in much of our later work.

Study guide

Section 1, the audio-tape section, gives you the necessary background in real integration. You should not spend too much time on this material — indeed, if you are already familiar with it, you can omit the section altogether.

The most important parts of this unit are Section 2 and Subsection 3.1. Since much of our work in the course depends on this material, it is important that you understand it before proceeding further.

Finally, it is also important that you understand the statement and use of the Estimation Theorem (Subsection 4.2), but you can omit its proof on a first reading.

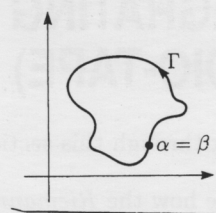


Figure 0.3

See Example 2.2 and Problem 3.5(a).

Here, the paths Γ_1 and Γ_2 are each circles traversed once anticlockwise using the standard parametrization.

1 INTEGRATING REAL FUNCTIONS (AUDIO-TAPE)

After working through this section, you should be able to:

- (a) appreciate how the *Riemann integral* $\int_a^b f(x) dx$, where f is a real function which is continuous on $[a, b]$, can be defined;
- (b) state the main properties of the Riemann integral.

In this section we remind you of the most important features of real integration. In particular, we outline the main properties of the *Riemann integral* of a continuous function.

Recall that a primary aim of real integration is to determine the *area under a curve*; that is, the area bounded by a graph $y = f(x)$, the x -axis, and the two vertical lines $x = a$ and $x = b$ (see Figure 1.1).

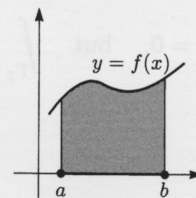


Figure 1.1

In order to define such an area formally, we first introduce the idea of a *partition*. We split the interval $[a, b]$ into a finite number of subintervals — this splitting is called a **partition** of $[a, b]$ (see Figure 1.2).

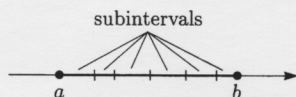


Figure 1.2 A partition

We can then underestimate the area under $y = f(x)$ by constructing the *largest* possible rectangles lying below the graph, with the various subintervals as bases, and adding their areas. Similarly, we can overestimate the area under $y = f(x)$ by constructing the *smallest* possible rectangles lying above the graph, with the same subintervals as bases, and adding their areas. These estimates are illustrated in Figures 1.3 and 1.4.

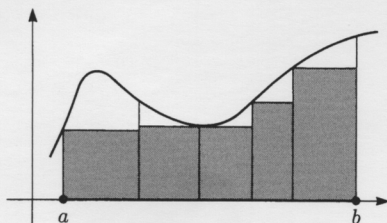


Figure 1.3 An underestimate

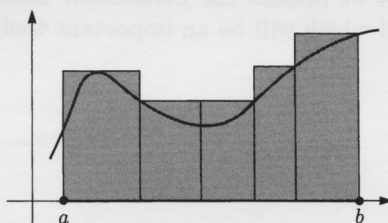


Figure 1.4 An overestimate

We now let the number of subintervals tend to infinity, in such a way that the lengths of the subintervals tend to zero. If the underestimates and overestimates tend to a common limit A , then we call A the *area under* $y = f(x)$, and write

$$A = \int_a^b f(x) dx.$$

Unfortunately, there is a snag when we try to generalize this approach to complex integrals. Inequalities between complex numbers have no meaning, and so the terms *largest* and *smallest*, and *above* and *below* are also meaningless.

We can get around this difficulty by modifying our approach to the area under a graph. To do this, we start with a partition of $[a, b]$, as before. We next choose any point inside each subinterval, and construct the rectangle whose base is the subinterval, and whose height is the value of the function at the chosen point. The sum of the areas of these rectangles is then an approximation to the area under the graph (see Figure 1.5).

This type of integral is named after Bernhard Riemann, who introduced it in 1854.

For definiteness, we shall choose the right-hand endpoint of each subinterval (as in Figure 1.5), although this is not essential.

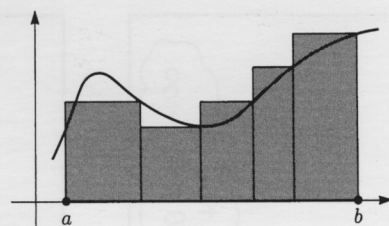


Figure 1.5 An approximation

We now let the number of subintervals tend to infinity, as before. If the sum of the areas of the constituent rectangles tends to a limit A , then we call A the *area under the graph*. This value agrees with that obtained by the earlier method and, as you will see in Section 2, this approach generalizes without difficulty to complex integrals.

In the following audio-tape section, we formally define the Riemann integral, as described above, and then summarize its main properties. We omit all proofs of general results.

The following sums will be helpful when you listen to the audio tape:

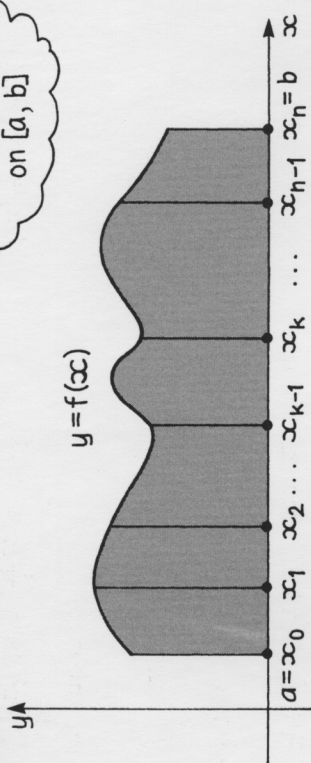
$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6};$$

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

NOW START THE TAPE.



1. Riemann sums



Partition: $P = \{[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]\}$

$\delta x_k = x_k - x_{k-1}$, $k = 1, 2, \dots, n$

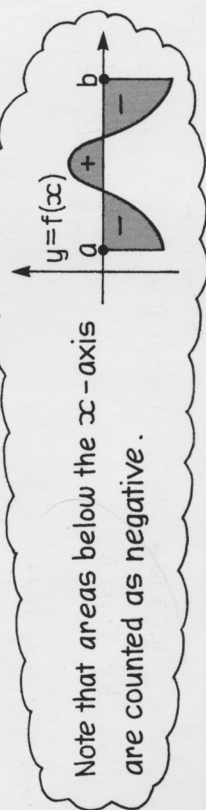
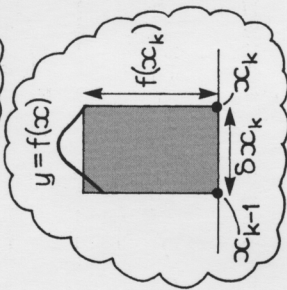
Mesh: $\|P\| = \max \{\delta x_1, \delta x_2, \dots, \delta x_n\}$

The 'area under $y = f(x)$ ' is approximately

$$R(f, P) = \sum_{k=1}^n f(x_k) \delta x_k,$$

a Riemann sum for f .

Intuitively, $R(f, P)$ converges, as $\|P\| \rightarrow 0$, to the 'area under $y = f(x)$ '.



Note that areas below the x -axis are counted as negative.

2. An example: $f(x) = x^2$ ($x \in [0, 1]$)

$$P_n = \left\{ \left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \dots, \left[\frac{n-1}{n}, 1\right] \right\}$$

n subintervals,
equal length

$$x_k = \frac{k}{n}, \quad k = 0, 1, 2, \dots, n$$

$$\delta x_k = \frac{1}{n}, \quad k = 1, 2, \dots, n$$

$$\|P_n\| = \frac{1}{n}$$

$$R(f, P_n) = \sum_{k=1}^n \left(\frac{k}{n}\right)^2 \frac{1}{n}$$

$$= \frac{1}{n^3} (1^2 + 2^2 + \dots + n^2)$$

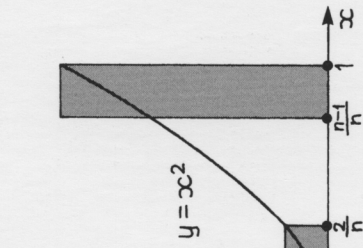
$$= \frac{n(n+1)(2n+1)}{6n^3}$$

$$= \frac{1}{6} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right).$$

$$\text{Hence } \lim_{n \rightarrow \infty} R(f, P_n) = \frac{1}{6} \times 1 \times 2 = \frac{1}{3}.$$

Note that

$$\|P_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$



3. Problem 1.1

With $f(x) = x^3$ ($x \in [0, 1]$) and P_n as in Frame 2, prove that

$$R(f, P_n) = \frac{1}{4} \left(1 + \frac{1}{n}\right)^2,$$

and determine $\lim_{n \rightarrow \infty} R(f, P_n)$.

4. Riemann integrals

f real-valued and continuous on $[a, b]$

Theorem There is a number A such that

$$\lim_{n \rightarrow \infty} R(f, P_n) = A,$$

whenever $\|P_n\| \rightarrow 0$.

The Riemann integral of f over $[a, b]$:

$$\int_a^b f(x) dx = A.$$

For example, by Frame 2,

$$\int_0^1 x^2 dx = \frac{1}{3}.$$

For $a > b$, define

$$\int_a^b f(x) dx = -\int_b^a f(x) dx;$$

$$\text{also } \int_a^a f(x) dx = 0.$$

A stands for 'area'.

5. Fundamental Theorem of Calculus

f real-valued and continuous on $[a, b]$

The function F is a primitive of f on $[a, b]$ if F is differentiable on $[a, b]$ and

$$F'(x) = f(x), \quad \text{for } x \in [a, b].$$

Theorem If F is a primitive of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a).$$

For example,

$$\int_0^1 x^2 dx = \left[\frac{1}{3} x^3 \right]_0^1 = \frac{1}{3}.$$

$$f(x) = x^2, \quad F(x) = \frac{1}{3} x^3$$

6. Properties of integrals

f, g real-valued and continuous on $[a, b]$

Identities

$$\text{Sum Rule } \int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

Combination Rules

$$\text{Multiple Rule } \int_a^b \lambda f(x) dx = \lambda \int_a^b f(x) dx, \quad \text{for } \lambda \in \mathbb{R}$$

$$\text{Additivity } \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$\text{Substitution } \int_a^b f(g(x)) g'(x) dx = \int_{g(a)}^{g(b)} f(t) dt$$

$t = g(x)$
 $dt = g'(x) dx$

$$\text{Parts } \int_a^b f'(x) g(x) dx = [f(x) g(x)]_a^b - \int_a^b f(x) g'(x) dx$$

$(f', g' \text{ continuous on } [a, b])$

Inequalities

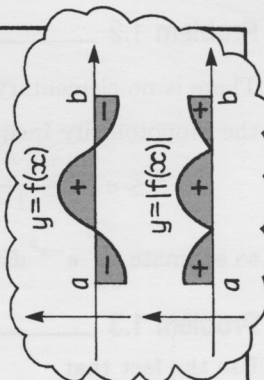
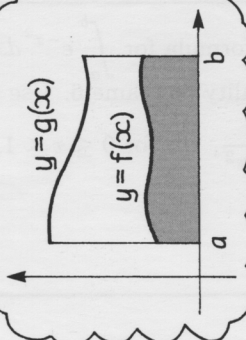
Monotonicity

If $f(x) \leq g(x)$, for $x \in [a, b]$, then

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

Triangle Inequality

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$$



Problem 1.2

There is no elementary formula for $\int_a^b e^{-x^2} dx$, but we can estimate it by using the Monotonicity Inequality of Frame 6. Use this approach and the fact that

$$e^{-x} \leq e^{-x^2} \leq \frac{1}{1+x^2}, \quad \text{for } 0 \leq x \leq 1,$$

to estimate $\int_0^1 e^{-x^2} dx$.

Problem 1.3

Use the fact that

$$\sin x \leq x, \quad \text{for } x \geq 0,$$

to show that

$$\left| \int_0^{\pi/4} \sin(-x) dx \right| \leq \frac{\pi^2}{32}.$$

Compare this estimate with the value of the integral.

2 INTEGRATING COMPLEX FUNCTIONS

After working through this section, you should be able to:

- define the *integral* $\int_{\Gamma} f(z) dz$, where Γ is a smooth path in \mathbb{C} and f is continuous on Γ , and hence evaluate such integrals;
- recognize *equivalent parametrizations*;
- explain what is meant by a *contour* Γ , define the (*contour*) *integral* of f along Γ , and evaluate such integrals;
- define the *reverse contour* of a given contour, and state and use the Reverse Contour Theorem.

2.1 Integration along a smooth path

We come now to the central idea of this unit — the integration of a continuous complex function f along a given smooth path Γ with initial point α and final point β . Our aim is to define the integral

$$\int_{\Gamma} f(z) dz.$$

There are two ways of achieving this aim. One method is to imitate the approach of Section 1:

- take a *partition* of the path Γ into subpaths

$$P = \{\Gamma_1, \Gamma_2, \dots, \Gamma_n\},$$

determined by points $\alpha = z_0, z_1, \dots, z_n = \beta$ (see Figures 2.1 and 2.2);

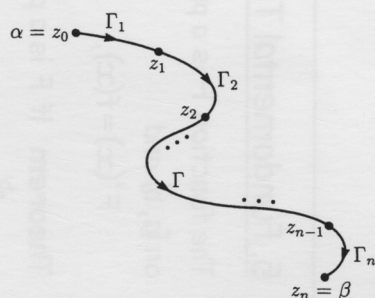


Figure 2.1 A partition of Γ

For example, $\int_{\Gamma} z^2 dz$, where Γ is the line segment from 0 to $1+i$.

You do not need to remember this method.

Frame 1

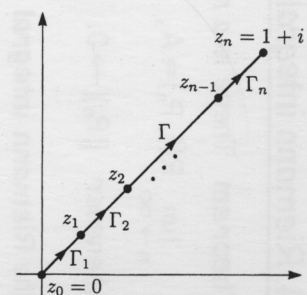


Figure 2.2 A partition of the line segment from 0 to $1+i$

- define a complex *Riemann sum*

$$R(f, P) = \sum_{k=1}^n f(z_k) \delta z_k,$$

Frame 1

where $\delta z_k = z_k - z_{k-1}$, for $k = 1, 2, \dots, n$;

- define the complex *Riemann integral* $\int_{\Gamma} f(z) dz$ to be

$$\lim_{n \rightarrow \infty} R(f, P_n),$$

Frame 4

where P_n is any sequence of partitions of Γ for which the mesh

$$\|P_n\| = \max\{|\delta z_k| : 1 \leq k \leq n\}$$

tends to 0 as n tends to ∞ .

It can be shown (although it is quite hard to do so) that this limit exists when f is continuous; thus the integral is well defined. We can then develop the standard properties of integrals, such as the Additivity Rule and the Combination Rules, by imitating the discussion of the real Riemann integral.

The other, quicker, method is to define a complex integral in terms of two real integrals. To do this we make use of the parametrization of the smooth path Γ , denoted as usual by

$$\gamma: t \mapsto \gamma(t) \quad (t \in [a, b]),$$

so that $\gamma(a) = \alpha$ and $\gamma(b) = \beta$. (Figure 2.3 shows this for the line segment from 0 to $1 + i$.) A typical Riemann sum for $\int_{\Gamma} f(z) dz$ is then of the form

$$R(f, P) = \sum_{k=1}^n f(z_k) \delta z_k,$$

where

$$z_k = \gamma(t_k), \quad k = 0, 1, \dots, n,$$

with

$$a = t_0 < t_1 < \dots < t_n = b,$$

as shown in Figure 2.4.

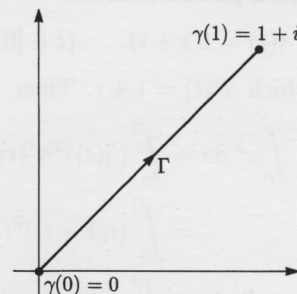


Figure 2.3

$$\gamma(t) = t(1 + i) \quad (t \in [0, 1])$$

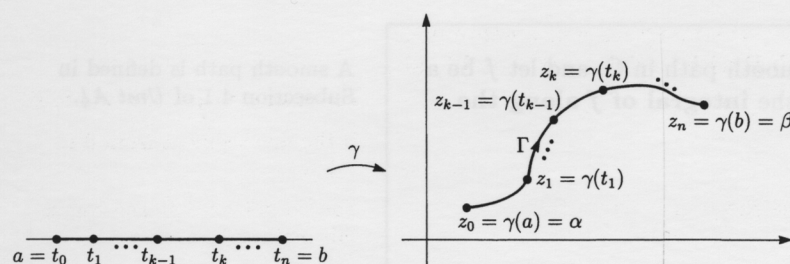


Figure 2.4

Now notice that, for $k = 1, 2, \dots, n$,

$$\begin{aligned} \delta z_k &= z_k - z_{k-1} \\ &= \gamma(t_k) - \gamma(t_{k-1}) \end{aligned}$$

and so, if t_k is close to t_{k-1} , then, to a good approximation,

$$\begin{aligned} \delta z_k &\cong \gamma'(t_k)(t_k - t_{k-1}) \\ &= \gamma'(t_k) \delta t_k, \quad \text{say.} \end{aligned}$$

Thus, if $\max\{\delta t_k : k = 1, 2, \dots, n\}$ is small, then, to a good approximation,

$$R(f, P) \cong \sum_{k=1}^n f(\gamma(t_k))\gamma'(t_k)\delta t_k.$$

Now the expression on the right is a Riemann sum for the integral

$$\int_a^b f(\gamma(t))\gamma'(t) dt. \quad (*)$$

The integrand

$$t \mapsto f(\gamma(t))\gamma'(t) \quad (t \in [a, b])$$

is a complex-valued function of a real variable, so the integral $(*)$ can be evaluated by splitting $f(\gamma(t))\gamma'(t)$ into its real and imaginary parts and then evaluating the resulting pair of real integrals. For example, to evaluate

$$\int_{\Gamma} z^2 dz,$$

where Γ is the line segment with initial point 0 and final point $1 + i$, we use the standard parametrization

$$\gamma(t) = t(1 + i) \quad (t \in [0, 1]),$$

for which $\gamma'(t) = 1 + i$. Then

$$\begin{aligned} \int_{\Gamma} z^2 dz &= \int_0^1 (\gamma(t))^2 \gamma'(t) dt \\ &= \int_0^1 (t(1 + i))^2 (1 + i) dt \\ &= \int_0^1 (-2 + 2i)t^2 dt \\ &= -2 \int_0^1 t^2 dt + 2i \int_0^1 t^2 dt \\ &= -2 \left[\frac{1}{3}t^3 \right]_0^1 + 2i \left[\frac{1}{3}t^3 \right]_0^1 \\ &= -\frac{2}{3} + \frac{2}{3}i. \end{aligned}$$

Here $f(z) = z^2$.

Here the two real integrals are the same, but in general they will be different.

We shall take $(*)$ to be our definition of $\int_{\Gamma} f(z) dz$.

Definition Let $\Gamma : \gamma(t)$ ($t \in [a, b]$) be a smooth path in \mathbb{C} , and let f be a function which is continuous on Γ . Then the **integral of f along the path Γ** , denoted by $\int_{\Gamma} f(z) dz$, is

$$\int_{\Gamma} f(z) dz = \int_a^b f(\gamma(t))\gamma'(t) dt.$$

The integral is evaluated by splitting $f(\gamma(t))\gamma'(t)$ into its real and imaginary parts and evaluating the resulting pair of real integrals,

$$\int_a^b f(\gamma(t))\gamma'(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt,$$

where $u(t) = \operatorname{Re}(f(\gamma(t))\gamma'(t))$ and $v(t) = \operatorname{Im}(f(\gamma(t))\gamma'(t))$.

A smooth path is defined in Subsection 4.1 of Unit A4.

Remarks

1 Since f is continuous on Γ and γ is a smooth parametrization, the functions

$$t \mapsto f(\gamma(t)) \quad \text{and} \quad t \mapsto \gamma'(t)$$

are both continuous on $[a, b]$, and so the function

$$t \mapsto f(\gamma(t))\gamma'(t)$$

is continuous on $[a, b]$. It follows that the real functions u and v are continuous on $[a, b]$, and hence $\int_a^b u(t) dt$ and $\int_a^b v(t) dt$ exist, and so $\int_a^b f(\gamma(t))\gamma'(t) dt$ exists.

2 An alternative notation for $\int_{\Gamma} f(z) dz$ is

$$\int_{\Gamma} f.$$

We shall sometimes find this notation convenient.

3 Since $\int_a^b f(\gamma(t))\gamma'(t) dt$ is defined in terms of two real integrals, the identities given in Frame 6 can be adapted to hold for integrals of complex-valued functions of a real variable.

4 If the path of integration Γ has a standard parametrization γ , then, unless otherwise stated, we use γ in the evaluation of $\int_{\Gamma} f.$ Unit A2, Section 2

5 To help to remember the formula used to define $\int_{\Gamma} f(z) dz$, notice its resemblance to the formula for integration by substitution

$$z = \gamma(t), \quad dz = \gamma'(t) dt.$$

The following examples illustrate the method of evaluation of integrals along paths.

Example 2.1

Evaluate $\int_{\Gamma} \bar{z} dz$, where Γ is the line segment from 0 to $1 + i$.

Solution

The standard parametrization is

$$\gamma(t) = t(1 + i) \quad (t \in [0, 1]).$$

Let $f(z) = \bar{z}$. Then

$$f(\gamma(t)) = \overline{t(1 + i)} = t(1 - i),$$

and since $\gamma'(t) = 1 + i$, we obtain

$$\begin{aligned} \int_{\Gamma} \bar{z} dz &= \int_0^1 t(1 - i)(1 + i) dt \\ &= \int_0^1 2t dt + i \int_0^1 0 dt \\ &= [t^2]_0^1 = 1. \quad \blacksquare \end{aligned}$$

In the next example, we use the observation in Remark 5 to set out the evaluation.

Example 2.2

Evaluate $\int_{\Gamma} \frac{1}{z} dz$, where Γ is the unit circle $\{z : |z| = 1\}$.

Solution

The standard parametrization is

$$\gamma(t) = e^{it} \quad (t \in [0, 2\pi]).$$

Then $z = e^{it}$, $1/z = e^{-it}$ and $dz = ie^{it} dt$. Hence

$$\gamma'(t) = ie^{it}$$

$$\begin{aligned} \int_{\Gamma} \frac{1}{z} dz &= \int_0^{2\pi} e^{-it} ie^{it} dt \\ &= i \int_0^{2\pi} 1 dt = 2\pi i. \quad \blacksquare \end{aligned}$$

In Examples 2.1 and 2.2, we specified the smooth path Γ in geometric terms and then used the standard parametrization in each case (since no parametrizations were given). The following problem suggests that choosing a different parametrization may lead to the same answer.

Problem 2.1

- (a) Verify that the result of Example 2.1 is unchanged if we use the smooth parametrization

$$\gamma(t) = 2t(1 + i) \quad (t \in [0, \tfrac{1}{2}]).$$

- (b) Verify that the result of Example 2.2 is unchanged if we use the smooth parametrization

$$\gamma(t) = e^{3it} \quad (t \in [0, \tfrac{2}{3}\pi]).$$

To explain why the parametrizations in Problem 2.1 give the same answers as Examples 2.1 and 2.2, we introduce the idea of *equivalent parametrizations*.

Definition Two parametrizations

$$\gamma_1 : [a_1, b_1] \longrightarrow \mathbb{C} \quad \text{and} \quad \gamma_2 : [a_2, b_2] \longrightarrow \mathbb{C}$$

are **equivalent** if there is a function $h : [a_1, b_1] \longrightarrow [a_2, b_2]$ satisfying

(a) $h(a_1) = a_2, h(b_1) = b_2,$

(b) h' exists on $[a_1, b_1]$, and is continuous and positive there,

such that

$$\gamma_1(t) = \gamma_2(h(t)), \quad \text{for } t \in [a_1, b_1].$$

This definition provides an equivalence relation on the set of such parametrizations. In particular, h^{-1} exists and $\gamma_2(s) = \gamma_1(h^{-1}(s))$, for $s \in [a_2, b_2]$.

Note that the images of γ_1 and γ_2 are the same set Γ , say, (see Figure 2.5) but that $\Gamma : \gamma_1(t)$ ($t \in [a_1, b_1]$) and $\Gamma : \gamma_2(t)$ ($t \in [a_2, b_2]$) are different, although **equivalent**, paths. These equivalent paths have the same initial points and the same final points.

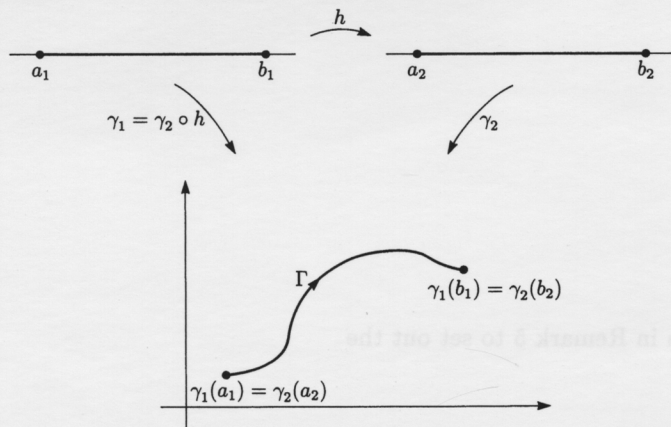


Figure 2.5

For example, the following parametrizations from Example 2.1 and Problem 2.1(a)

$$\gamma_1(t) = t(1 + i) \quad (t \in [0, 1]),$$

$$\gamma_2(t) = 2t(1 + i) \quad (t \in [0, \tfrac{1}{2}]),$$

are equivalent, with

$$[a_1, b_1] = [0, 1], \quad [a_2, b_2] = [0, \tfrac{1}{2}] \quad \text{and} \quad h(t) = \tfrac{1}{2}t \quad (t \in [0, 1]),$$

since $h(0) = 0, h(1) = \tfrac{1}{2}, h'(t) = \tfrac{1}{2}$ is continuous and positive on $[0, 1]$, and

$$\gamma_2(h(t)) = 2h(t)(1 + i) = t(1 + i) = \gamma_1(t), \quad \text{for } 0 \leq t \leq 1.$$

Theorem 2.1 If the smooth paths $\Gamma : \gamma_1(t)$ ($t \in [a_1, b_1]$) and $\Gamma : \gamma_2(t)$ ($t \in [a_2, b_2]$) are equivalent and the function f is continuous on Γ , then $\int_{\Gamma} f(z) dz$ does not depend on which parametrization, γ_1 or γ_2 , is used.

Proof If $\gamma_1(t) = \gamma_2(h(t))$, for $t \in [a_1, b_1]$, where h satisfies the hypotheses of the above definition, then

$$\begin{aligned} \int_{a_1}^{b_1} f(\gamma_1(t)) \gamma_1'(t) dt &= \int_{a_1}^{b_1} f(\gamma_2(h(t))) \gamma_2'(h(t)) h'(t) dt \quad (\text{Chain Rule}) \\ &= \int_{h(a_1)}^{h(b_1)} f(\gamma_2(s)) \gamma_2'(s) ds \\ &= \int_{a_2}^{b_2} f(\gamma_2(s)) \gamma_2'(s) ds, \end{aligned}$$

as required. ■

In practical terms, this theorem allows you to choose any convenient equivalent (smooth) parametrization when evaluating a complex integral along a given path $\Gamma : \gamma$. In the next subsection, we shall see how this can be helpful.

Problem 2.2

Which of the following pairs of smooth paths Γ_1 and Γ_2 are equivalent? Justify your answers.

- (a) $\Gamma_1 : \gamma_1(t) = 2 + i(t - 1)$ ($t \in [1, 2]$); $\Gamma_2 : \gamma_2(t) = 2 + it$ ($t \in [0, 1]$).
 (b) $\Gamma_1 : \gamma_1(t) = (1 - t)i + t$ ($t \in [0, 1]$); $\Gamma_2 : \gamma_2(t) = (1 - t) + ti$ ($t \in [0, 1]$).

For further practice in integration, try the following problem.

Problem 2.3

Evaluate the following integrals.

- (a) $\int_{\Gamma} \operatorname{Re} z \, dz$, where Γ is the line segment from 0 to $1 + 2i$.
 (b) $\int_{\Gamma} \frac{1}{(z - \alpha)^2} \, dz$, where Γ is the circle with centre α and radius r .

2.2 Integration along a contour

Consider the path Γ from 0 to i in Figure 2.6, with parametrization $\gamma : [0, 3] \rightarrow \mathbb{C}$ given by

$$\gamma(t) = \begin{cases} 2t, & 0 \leq t \leq 1, \\ 2 + i(t - 1), & 1 \leq t \leq 2, \\ 2 + i - 2(t - 2), & 2 \leq t \leq 3. \end{cases}$$

This path is not smooth, because of difficulties at the corners 2 and $2 + i$. However, Γ can be split into three smooth straight-line paths, joined end to end. This leads to the idea of a *contour*: it is simply what we get when we place a finite number of smooth paths end to end.

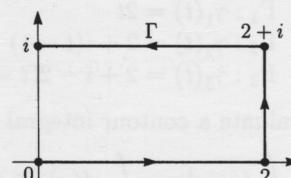


Figure 2.6

Definitions A contour Γ is a path which can be subdivided into a finite number of smooth paths $\Gamma_1, \Gamma_2, \dots, \Gamma_n$, joined end to end; the order of these constituent smooth paths is indicated by writing

$$\Gamma = \Gamma_1 + \Gamma_2 + \dots + \Gamma_n.$$

The **initial point** of Γ is the initial point of Γ_1 , and the **final point** of Γ is the final point of Γ_n . (See Figure 2.7.)

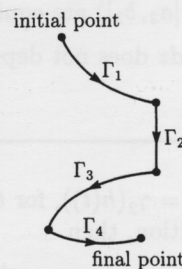


Figure 2.7
 $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$

For example, the contour Γ in Figure 2.6 can be written as $\Gamma_1 + \Gamma_2 + \Gamma_3$, where Γ_1, Γ_2 and Γ_3 are smooth paths with smooth parametrizations

$$\begin{aligned}\gamma_1(t) &= 2t & (t \in [0, 1]), \\ \gamma_2(t) &= 2 + i(t - 1) & (t \in [1, 2]), \\ \gamma_3(t) &= 2 + i - 2(t - 2) & (t \in [2, 3]).\end{aligned}$$

We saw above how to integrate a continuous function along a smooth path. It is natural to extend this definition to contours, by splitting the contour into smooth paths and integrating along each in turn. We formalize this idea in the following definition.

Definition Let $\Gamma = \Gamma_1 + \Gamma_2 + \dots + \Gamma_n$ be a contour, and let f be a function which is continuous on Γ . Then the **(contour) integral of f along Γ** , denoted by $\int_{\Gamma} f(z) dz$, is

$$\int_{\Gamma} f(z) dz = \int_{\Gamma_1} f(z) dz + \int_{\Gamma_2} f(z) dz + \dots + \int_{\Gamma_n} f(z) dz.$$

Remarks

1 It is clear that a contour can be split into smooth paths in many different ways. Fortunately, all such splittings lead to the same value for the contour integral. We omit the proof of this result, as it is straightforward but tedious.

2 When evaluating an integral along a contour $\Gamma = \Gamma_1 + \Gamma_2 + \dots + \Gamma_n$, we often consider each smooth path $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ separately, using a convenient parametrization in each case. This is permissible by Theorem 2.1. For example, consider the contour $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ (of Figure 2.6) given by

$$\begin{aligned}\Gamma_1 : \gamma_1(t) &= 2t & (t \in [0, 1]), \\ \Gamma_2 : \gamma_2(t) &= 2 + i(t - 1) & (t \in [1, 2]), \\ \Gamma_3 : \gamma_3(t) &= 2 + i - 2(t - 2) & (t \in [2, 3]).\end{aligned}$$

To evaluate a contour integral of the form

$$\int_{\Gamma} f(z) dz = \int_{\Gamma_1} f(z) dz + \int_{\Gamma_2} f(z) dz + \int_{\Gamma_3} f(z) dz,$$

it would be convenient and permissible to use the equivalent parametrizations

$$\begin{aligned}\gamma_1(t) &= t & (t \in [0, 2]), \\ \gamma_2(t) &= 2 + it & (t \in [0, 1]), \\ \gamma_3(t) &= 2 + i - t & (t \in [0, 2]).\end{aligned}$$

3 The alternative notation $\int_{\Gamma} f$ is also used for contour integrals.

Example 2.3

Evaluate $\int_{\Gamma} z^2 dz$, where Γ is the contour shown in Figure 2.8.

Solution

We split Γ into two smooth paths: $\Gamma = \Gamma_1 + \Gamma_2$, where Γ_1 is the line segment from 0 to 1 with parametrization $\gamma_1(t) = t$ ($t \in [0, 1]$), and Γ_2 is the line segment from 1 to $1 + i$, with parametrization $\gamma_2(t) = 1 + it$ ($t \in [0, 1]$). Then

$$\begin{aligned} \int_{\Gamma} z^2 dz &= \int_{\Gamma_1} z^2 dz + \int_{\Gamma_2} z^2 dz \\ &= \int_0^1 t^2 dt + \int_0^1 (1 + it)^2 i dt \\ &= \int_0^1 t^2 dt + \int_0^1 (-2t + i - it^2) dt \\ &= \int_0^1 t^2 dt + \int_0^1 (-2t) dt + i \int_0^1 (1 - t^2) dt \\ &= \left[\frac{1}{3} t^3 \right]_0^1 + \left[-t^2 \right]_0^1 + i \left[t - \frac{1}{3} t^3 \right]_0^1 \\ &= \frac{1}{3} - 1 + i \left(1 - \frac{1}{3} \right) = -\frac{2}{3} + \frac{2}{3} i. \quad \blacksquare \end{aligned}$$

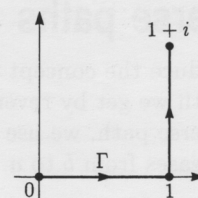


Figure 2.8

Note that we have omitted the initial step, involving f and γ or z and dz , in this integration.

Notice that this answer is the same as that obtained earlier for $\int_{\Gamma} z^2 dz$, where Γ is the line segment from 0 to $1 + i$. The reason for this will be given in Section 3.

Problem 2.4

Evaluate $\int_{\Gamma} \bar{z} dz$ for each of the following contours Γ .

(a)

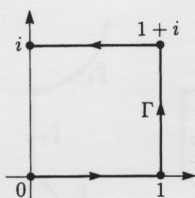


Figure 2.9

(b)

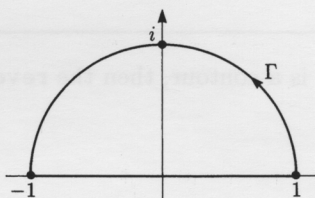


Figure 2.10

In part (b) the contour consists of a line segment and a semi-circle, traversed once anticlockwise. Take -1 to be the initial (and final) point of the contour.

We conclude this subsection by stating the Combination Rules for contour integrals. To prove them, we split the contour Γ into constituent smooth paths, and use the Combination Rules for real integration to prove the results for each path. We omit the details.

Frame 6

Theorem 2.2 Combination Rules

Let Γ be a contour, and let f and g be functions which are continuous on Γ . Then

Sum Rule $\int_{\Gamma} (f(z) + g(z)) dz = \int_{\Gamma} f(z) dz + \int_{\Gamma} g(z) dz;$

Multiple Rule $\int_{\Gamma} \lambda f(z) dz = \lambda \int_{\Gamma} f(z) dz, \text{ where } \lambda \in \mathbb{C}.$

2.3 Reverse paths and contours

We now introduce the concept of the *reverse path* of a smooth path Γ : this is simply the path we get by reversing the direction of the arrow on Γ . In order to define the reverse path, we use the fact that as t increases from a to b , so $a + b - t$ decreases from b to a .

Definition Let $\Gamma : \gamma(t)$ ($t \in [a, b]$) be a smooth path. Then the **reverse path**, denoted by $\tilde{\Gamma}$, is the path with parametrization $\tilde{\gamma}(t)$, where

$$\tilde{\gamma}(t) = \gamma(a + b - t) \quad (t \in [a, b]).$$

Some texts use the name *opposite path*.

Note that the initial point $\tilde{\gamma}(a)$ of $\tilde{\Gamma}$ is the final point $\gamma(b)$ of Γ , and that the final point $\tilde{\gamma}(b)$ of $\tilde{\Gamma}$ is the initial point $\gamma(a)$ of Γ (see Figure 2.11). Also note that, as sets, Γ and $\tilde{\Gamma}$ are the same, and that $\tilde{\Gamma} : \tilde{\gamma}$ is smooth.

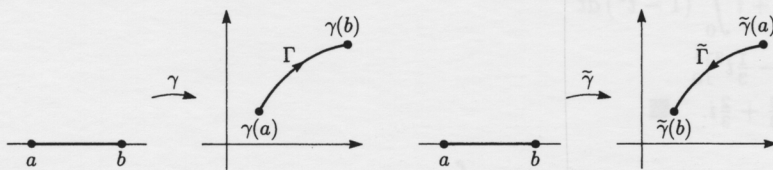


Figure 2.11 A smooth path Γ and its reverse path $\tilde{\Gamma}$

Problem 2.5

Write down the reverse path of the path Γ with parametrization

$$\gamma(t) = 2 + i - t \quad (t \in [0, 2]).$$

We can also define a *reverse contour*. This is done in the natural way — namely, by reversing each of the smooth constituent paths and reversing the order in which they are traversed.

Definition If $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$ is a contour, then the **reverse contour** $\tilde{\Gamma}$ of Γ is defined by

$$\tilde{\Gamma} = \tilde{\Gamma}_n + \tilde{\Gamma}_{n-1} + \cdots + \tilde{\Gamma}_1.$$

(See Figure 2.12.)

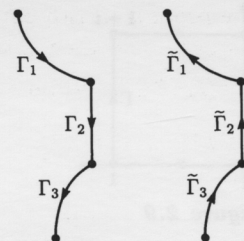


Figure 2.12

For example, if $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ is the contour from 0 to i in Figure 2.13, with smooth parametrizations

$$\begin{aligned} \gamma_1(t) &= t & (t \in [0, 2]), \\ \gamma_2(t) &= 2 + it & (t \in [0, 1]), \\ \gamma_3(t) &= 2 + i - t & (t \in [0, 2]), \end{aligned}$$

then $\tilde{\Gamma} = \tilde{\Gamma}_3 + \tilde{\Gamma}_2 + \tilde{\Gamma}_1$ is the contour from i to 0 in Figure 2.14, with smooth parametrizations

$$\begin{aligned} \tilde{\gamma}_3(t) &= t + i & (t \in [0, 2]), \\ \tilde{\gamma}_2(t) &= 2 + i(1 - t) & (t \in [0, 1]), \\ \tilde{\gamma}_1(t) &= 2 - t & (t \in [0, 2]). \end{aligned}$$

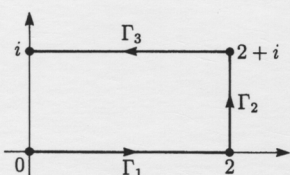


Figure 2.13 $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$

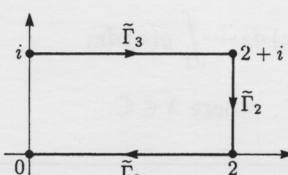


Figure 2.14 $\tilde{\Gamma} = \tilde{\Gamma}_3 + \tilde{\Gamma}_2 + \tilde{\Gamma}_1$

Example 2.4

Evaluate $\int_{\tilde{\Gamma}} \bar{z} dz$, where $\tilde{\Gamma}$ is the reverse path of the line segment Γ from 0 to $1 + i$, and compare your answer with that of Example 2.1.

Solution

In Example 2.1, we used the smooth parametrization

$$\gamma(t) = t(1 + i) \quad (t \in [0, 1]).$$

For the reverse path $\tilde{\Gamma}$, we need the parametrization

$$\tilde{\gamma}(t) = \gamma(1 - t) = (1 - t)(1 + i) \quad (t \in [0, 1]).$$

Then $\tilde{\gamma}'(t) = -(1 + i)$, and so

$$\begin{aligned} \int_{\tilde{\Gamma}} \bar{z} dz &= - \int_0^1 ((1 - t)(1 - i))(1 + i) dt \\ &= - \int_0^1 2(1 - t) dt \\ &= - [2t - t^2]_0^1 = -1. \quad \blacksquare \end{aligned}$$

$$\begin{aligned} z &= (1 - t)(1 + i), \\ \bar{z} &= (1 - t)(1 - i), \\ dz &= -(1 + i) dt. \end{aligned}$$

This example illustrates the general result that if we integrate a function along a reverse contour $\tilde{\Gamma}$, then the answer is the negative of the integral of the function along Γ .

Theorem 2.3 Reverse Contour Theorem

Let Γ be a contour, and let f be a function which is continuous on Γ . Then, if $\tilde{\Gamma}$ is the reverse contour of Γ ,

$$\int_{\tilde{\Gamma}} f(z) dz = - \int_{\Gamma} f(z) dz.$$

Proof The proof is in two parts. We first prove the result in the case when Γ is a smooth path, and then extend the proof to contours.

(a) Let $\Gamma: \gamma(t)$ ($t \in [a, b]$) be a smooth path. Then the parametrization of $\tilde{\Gamma}$ is

$$\tilde{\gamma}(t) = \gamma(a + b - t) \quad (t \in [a, b]),$$

so that

$$\begin{aligned} \int_{\tilde{\Gamma}} f(z) dz &= \int_a^b f(\tilde{\gamma}(t)) \tilde{\gamma}'(t) dt \\ &= \int_a^b f(\gamma(a + b - t)) (-\gamma'(a + b - t)) dt \quad (\text{Chain Rule}) \\ &= \int_b^a f(\gamma(s)) \gamma'(s) ds \\ &= - \int_{\Gamma} f(z) dz. \end{aligned}$$

This use of the real substitution $s = a + b - t$, $ds = -dt$ can be justified by splitting the integrand into its real and imaginary parts.

(b) To extend the proof to a general contour Γ , we argue as follows.

Let $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$. Then $\tilde{\Gamma} = \tilde{\Gamma}_n + \tilde{\Gamma}_{n-1} + \cdots + \tilde{\Gamma}_1$, and we have

$$\begin{aligned} \int_{\tilde{\Gamma}} f &= \int_{\tilde{\Gamma}_n} f + \int_{\tilde{\Gamma}_{n-1}} f + \cdots + \int_{\tilde{\Gamma}_1} f \\ &= - \int_{\Gamma_n} f - \int_{\Gamma_{n-1}} f - \cdots - \int_{\Gamma_1} f \quad (\text{by part (a)}) \\ &= - \left(\int_{\Gamma_n} f + \int_{\Gamma_{n-1}} f + \cdots + \int_{\Gamma_1} f \right) \\ &= - \int_{\Gamma} f. \quad \blacksquare \end{aligned}$$

Problem 2.6

Using the results of Examples 2.2 and 2.3, verify the result of Theorem 2.3 in the following cases.

- (a) $f(z) = 1/z$, and Γ is the unit circle $\{z : |z| = 1\}$.
- (b) $f(z) = z^2$, and Γ is the contour shown in Figure 2.15.

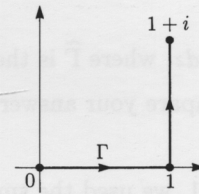


Figure 2.15

3 EVALUATING CONTOUR INTEGRALS

After working through this section, you should be able to:

- (a) state and use the Fundamental Theorem of Calculus for Contour Integrals;
- (b) state and use the Contour Independence Theorem;
- (c) use the technique of Integration by Parts;
- (d) state and use the Closed Contour Theorem, the Grid Path Theorem, the Zero Derivative Theorem and the Paving Theorem.

3.1 The Fundamental Theorem of Calculus

In Example 2.3, we saw that

$$\int_{\Gamma} z^2 dz = -\frac{2}{3} + \frac{2}{3}i,$$

where Γ is the contour shown in Figure 3.1. Our method was to write down a smooth parametrization for each of the two line segments, replace z in the integral by these parametrizations, and then integrate. It is, however, very tempting to write (as you would for a corresponding real integral)

$$\begin{aligned} \int_{\Gamma} z^2 dz &= \left[\frac{1}{3} z^3 \right]_0^{1+i} \\ &= \frac{1}{3}(1+i)^3 - \frac{1}{3}(0)^3 = -\frac{2}{3} + \frac{2}{3}i. \end{aligned}$$

The Fundamental Theorem of Calculus for contour integrals tells us that this method of evaluation is permissible under certain conditions. Before stating it, we need to extend the idea of a *primitive* from real to complex functions.

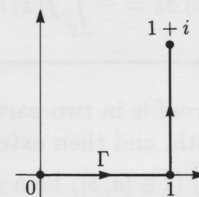


Figure 3.1

$$\begin{aligned} (1+i)^3 &= 1 + 3i + 3i^2 + i^3 \\ &= -2 + 2i. \end{aligned}$$

Definition Let f and F be functions defined on a region \mathcal{R} . Then F is a **primitive of f on \mathcal{R}** if F is analytic on \mathcal{R} and

$$F'(z) = f(z), \quad \text{for all } z \in \mathcal{R}.$$

F is also called an *antiderivative* or *indefinite integral* of f on \mathcal{R} .

For example, $F(z) = \frac{1}{3}z^3$ is a primitive of $f(z) = z^2$ on \mathbb{C} , since F is analytic on \mathbb{C} and $F'(z) = z^2$, for all $z \in \mathbb{C}$. Another primitive is $F(z) = \frac{1}{3}z^3 + 2i$; indeed, any function of the form $F(z) = \frac{1}{3}z^3 + c$, where $c \in \mathbb{C}$, is a primitive of f on \mathbb{C} .

Problem 3.1

Write down a primitive F for each of the following functions f on the given region \mathcal{R} .

- (a) $f(z) = e^{3iz}$, $\mathcal{R} = \mathbb{C}$
- (b) $f(z) = (1 + iz)^{-2}$, $\mathcal{R} = \mathbb{C} - \{i\}$
- (c) $f(z) = z^{-1}$, $\mathcal{R} = \{z : \operatorname{Re} z > 0\}$

The precise statement of the Fundamental Theorem of Calculus for contour integrals is as follows.

Theorem 3.1 Fundamental Theorem of Calculus

Let the function f be continuous on a region \mathcal{R} , let F be a primitive of f on \mathcal{R} , and let Γ be a contour in \mathcal{R} with initial point α and final point β .

Then

$$\int_{\Gamma} f(z) dz = F(\beta) - F(\alpha) (= [F(z)]_{\alpha}^{\beta}).$$

The proof is given later in this subsection.

In some texts, $[F(z)]_{\alpha}^{\beta}$ is written as $F(z)|_{\alpha}^{\beta}$.

For example, if $f(z) = z^2$, then f is continuous on \mathbb{C} and has a primitive $F(z) = \frac{1}{3}z^3$ there. Hence, for the contour Γ in Figure 3.1, we can write

$$\int_{\Gamma} z^2 dz = \left[\frac{1}{3}z^3\right]_0^{1+i} = \frac{1}{3}(1+i)^3 - \frac{1}{3}(0^3) = -\frac{2}{3} + \frac{2}{3}i.$$

Problem 3.2

Use the Fundamental Theorem of Calculus to evaluate

$$\int_{\Gamma} e^{3iz} dz,$$

where Γ is the semi-circular path shown in Figure 3.2.

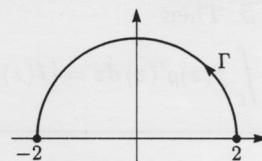


Figure 3.2

The following result is a simple, but important, consequence of the Fundamental Theorem of Calculus.

Theorem 3.2 Contour Independence Theorem

Let the function f be continuous and have a primitive F on a region \mathcal{R} , and let Γ_1 and Γ_2 be contours in \mathcal{R} with the same initial point α and the same final point β . Then

$$\int_{\Gamma_1} f(z) dz = \int_{\Gamma_2} f(z) dz.$$

You have seen that

$$\int_{\Gamma} z^2 dz = -\frac{2}{3} + \frac{2}{3}i,$$

where Γ is the contour with endpoints 0 and $1+i$ in Figure 3.1, and where Γ is the line segment from 0 to $1+i$ (see page 12).

Proof By the Fundamental Theorem of Calculus for contour integrals, the value of each of these integrals is $F(\beta) - F(\alpha)$. ■

The idea that a contour integral may, under suitable hypotheses, depend only on the endpoints of the contour (and not on the contour itself) will prove to have great significance in the next unit.

Problem 3.3

Use the Fundamental Theorem of Calculus to evaluate the following integrals.

- (a) $\int_{\Gamma} e^{-\pi z} dz$, where Γ is any contour from $-i$ to i .
- (b) $\int_{\Gamma} (3z - 1)^4 dz$, where Γ is any contour from 2 to $2i + \frac{1}{3}$.
- (c) $\int_{\Gamma} \sinh z dz$, where Γ is any contour from i to 1 .
- (d) $\int_{\Gamma} e^{\sin z} \cos z dz$, where Γ is any contour from 0 to $\frac{1}{2}\pi$.
- (e) $\int_{\Gamma} \frac{\sin z}{\cos^2 z} dz$, where Γ is any contour from 0 to π lying in $\mathbb{C} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$.

Parts (d) and (e) of Problem 3.3 were solved by noting that each integrand is of the form

$$g'(f(z))f'(z) = (g \circ f)'(z);$$

that is, by a form of *integration by substitution*. There is also a complex version of *integration by parts*, which we shall need later in the course.

Theorem 3.3 Integration by Parts

Let the functions f and g be analytic on a region \mathcal{R} , and let f' and g' be continuous on \mathcal{R} . Let Γ be a contour in \mathcal{R} with initial point α and final point β . Then

$$\int_{\Gamma} f(z)g'(z) dz = [f(z)g(z)]_{\alpha}^{\beta} - \int_{\Gamma} f'(z)g(z) dz.$$

Recall that $[f(z)g(z)]_{\alpha}^{\beta}$ means $f(\beta)g(\beta) - f(\alpha)g(\alpha)$.

Proof Let $H(z) = f(z)g(z)$ and $h(z) = f'(z)g(z) + f(z)g'(z)$. Then h is continuous on \mathcal{R} , by hypothesis. Also, h has primitive H , since H is analytic on \mathcal{R} and

$$H'(z) = h(z).$$

It follows from the Fundamental Theorem of Calculus that

$$\int_{\Gamma} h(z) dz = [H(z)]_{\alpha}^{\beta};$$

that is,

$$\int_{\Gamma} (f'(z)g(z) + f(z)g'(z)) dz = [f(z)g(z)]_{\alpha}^{\beta}.$$

Using the Sum Rule (Theorem 2.2) and rearranging the resulting equation, we obtain

$$\int_{\Gamma} f(z)g'(z) dz = [f(z)g(z)]_{\alpha}^{\beta} - \int_{\Gamma} f'(z)g(z) dz,$$

as required. ■

Example 3.1

Use integration by parts to evaluate

$$\int_{\Gamma} ze^{2z} dz,$$

where Γ is any contour from 0 to πi .

Solution

We take $f(z) = z$, $g(z) = \frac{1}{2}e^{2z}$, and $\mathcal{R} = \mathbb{C}$. Then f and g are analytic on \mathcal{R} , and $f'(z) = 1$ and $g'(z) = e^{2z}$ are continuous on \mathcal{R} .

Integrating by parts, we obtain

$$\begin{aligned}\int_{\Gamma} z e^{2z} dz &= \left[z \cdot \frac{1}{2} e^{2z} \right]_0^{\pi i} - \int_{\Gamma} 1 \cdot \left(\frac{1}{2} e^{2z} \right) dz \\ &= \left(\pi i \cdot \frac{1}{2} e^{2\pi i} - 0 \right) - \left[\frac{1}{4} e^{2z} \right]_0^{\pi i} \\ &= \frac{1}{2} \pi i - \left(\frac{1}{4} - \frac{1}{4} \right) = \frac{1}{2} \pi i. \quad \blacksquare\end{aligned}$$

Problem 3.4

Use integration by parts to evaluate the following integrals.

- (a) $\int_{\Gamma} z \cosh z dz$, where Γ is any contour from 0 to πi .
- (b) $\int_{\Gamma} \operatorname{Log} z dz$, where Γ is any contour from 1 to i lying in the region $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$.
(Hint: Take $f(z) = \operatorname{Log} z$ and $g(z) = z$.)

The Fundamental Theorem of Calculus is a very useful tool when the function f being integrated has an easily-determined primitive F . However, if the function f has no primitive, or if we are unable to find one, then we have to resort to the definition of an integral and use parametrization. For example, we cannot use the Fundamental Theorem of Calculus to evaluate

$$\int_{\Gamma} \bar{z} dz$$

along any contour, since the function $f(z) = \bar{z}$ has no primitive on any region. Indeed, it turns out that *if f is not differentiable, then f has no primitive F* . This is because, as we shall see in the next unit, any differentiable complex function can be differentiated as many times as we like. Thus, if f did have a primitive F , then F would be differentiable, with $F' = f$. Hence f would be differentiable, which is a contradiction. It follows that we cannot use the Fundamental Theorem of Calculus to evaluate integrals of non-differentiable functions such as

$$z \mapsto \bar{z}, \quad z \mapsto \operatorname{Re} z, \quad z \mapsto \operatorname{Im} z \quad \text{and} \quad z \mapsto |z|.$$

We conclude this subsection by proving the Fundamental Theorem of Calculus.

Proof of the Fundamental Theorem of Calculus

The proof is in two parts. We first prove the result in the case when Γ is a smooth path, and then extend the proof to contours.

- (a) Let $\Gamma : \gamma(t)$ ($t \in [a, b]$) be a smooth path. Then

$$\begin{aligned}\int_{\Gamma} f(z) dz &= \int_a^b f(\gamma(t)) \gamma'(t) dt \\ &= \int_a^b F'(\gamma(t)) \gamma'(t) dt \quad (\text{since } F' = f) \\ &= \int_a^b (F \circ \gamma)'(t) dt \quad (\text{Chain Rule}) \\ &= F(\gamma(b)) - F(\gamma(a)) \\ &= F(\beta) - F(\alpha) \quad (\text{since } \beta = \gamma(b), \alpha = \gamma(a)).\end{aligned}$$

To justify this step, split $F(\gamma(t))$ into real and imaginary parts and then use the Fundamental Theorem of Calculus for real integrals (Section 1, Frame 5).

- (b) To extend the proof to a general contour Γ with initial point α and final point β , we argue as follows.

Let $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$, and let the initial and final points of Γ_k be α_k and β_k , for $k = 1, 2, \dots, n$. Then

$$\alpha_1 = \alpha, \quad \alpha_2 = \beta_1, \quad \dots, \quad \alpha_n = \beta_{n-1}, \quad \beta_n = \beta. \quad (*)$$

Thus, by the above result for smooth paths, we have

$$\begin{aligned} \int_{\Gamma} f(z) dz &= \int_{\Gamma_1} f(z) dz + \int_{\Gamma_2} f(z) dz + \cdots + \int_{\Gamma_n} f(z) dz \\ &= (F(\beta_1) - F(\alpha_1)) + (F(\beta_2) - F(\alpha_2)) + \cdots + (F(\beta_n) - F(\alpha_n)) \\ &= F(\beta) - F(\alpha) \quad (\text{by } (*)). \quad \blacksquare \end{aligned}$$

3.2 Closed paths and grid paths

The Fundamental Theorem of Calculus has a number of important theoretical consequences, which we now describe. The first concerns integration around *closed contours*.

Definition A path or contour Γ is **closed** if its initial and final points coincide.

This use of the adjective 'closed' is different from that in *closed set*. The context will always tell you which meaning is appropriate.

For example, any circle is a closed contour, as are the contours in Figure 3.3.

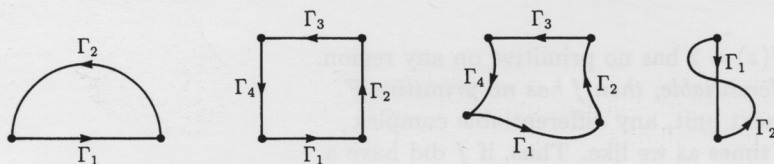


Figure 3.3 Closed contours

We remarked earlier (page 16) that any method of splitting a contour Γ into constituent smooth paths leads to the same value for a given contour integral along Γ . Using this fact, it can be shown that if Γ is a closed contour, then the value of any contour integral along Γ does not depend on the choice of initial (= final) point.

The following result about closed contours is a forerunner of Cauchy's Theorem — a major result which appears in the next unit.

Theorem 3.4 Closed Contour Theorem

Let the function f be continuous and have a primitive F on a region \mathcal{R} . Then

$$\int_{\Gamma} f(z) dz = 0,$$

for any closed contour Γ in \mathcal{R} .

Proof Since Γ is closed, its initial and final points α and β are equal. It follows from the Fundamental Theorem of Calculus that

$$\int_{\Gamma} f(z) dz = F(\beta) - F(\alpha) = 0. \quad \blacksquare$$

Use the Closed Contour Theorem to prove the following.

- (a) $\int_{\Gamma} 1/z \, dz = 0$, where Γ is the circle with centre $1 + i$, and radius 1.
 (b) $\int_{\Gamma} 1/z^2 \, dz = 0$, where Γ is the unit circle $\{z : |z| = 1\}$.

Next we use the Fundamental Theorem of Calculus to prove that an analytic function with zero derivative must be constant, if its domain is a region. To prove this result in its most general form, we need a further property of regions. Recall that, if \mathcal{R} is a region, then any two points in \mathcal{R} can be joined by a path in \mathcal{R} . In fact, such a path can always be chosen to be a contour of the following simple type (along which one can integrate).

Definition A **grid path** is a contour each of whose constituent smooth paths is a line segment parallel to either the real axis or the imaginary axis.

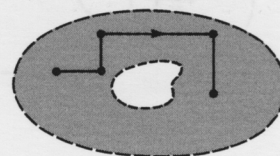


Figure 3.4

For example, the contours in Figures 2.6, 2.8, 2.9 are all grid paths, as is that shown in Figure 3.4.

Theorem 3.5 Grid Path Theorem

If \mathcal{R} is a region, then any two points in \mathcal{R} can be joined by a grid path in \mathcal{R} .

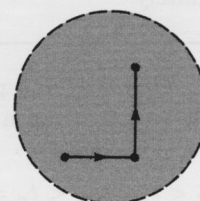


Figure 3.5

The assertion of the Grid Path Theorem is obvious if \mathcal{R} is an open disc since at most two line segments are required (Figure 3.5), but it is quite hard to prove for a *general* region. We postpone that proof till after the next result, which demonstrates how useful grid paths can be.

Theorem 3.6 Zero Derivative Theorem

Let the function F be analytic on a region \mathcal{R} , and let $F'(z) = 0$, for all z in \mathcal{R} . Then F is constant on \mathcal{R} .

Proof Let α and β be any two points in \mathcal{R} , and let Γ be any grid path in \mathcal{R} with initial point α and final point β . Since $F'(z) = 0$, for all z in \mathcal{R} , it follows from the Fundamental Theorem of Calculus that

$$\begin{aligned} F(\beta) - F(\alpha) &= \int_{\Gamma} F'(z) dz \\ &= \int_{\Gamma} 0 dz = 0. \end{aligned}$$

Thus $F(\alpha) = F(\beta)$, and hence F is constant on \mathcal{R} . \blacksquare

The next problem gives an important consequence of the Zero Derivative Theorem, which will be needed later in the course.

Problem 3.6

Prove that, if F_1 and F_2 are both primitives of a function f on a region \mathcal{R} , then

$$F_1(z) = F_2(z) + c, \quad \text{for all } z \text{ in } \mathcal{R},$$

where c is a complex constant.

3.3 Proof of the Grid Path Theorem

We have remarked that the assertion of the Grid Path Theorem is obvious if the region \mathcal{R} is an open disc (see Figure 3.5). To prove the theorem in general, we make use of this special case and show that we can construct a grid path joining two points α and β in \mathcal{R} , of the form

This subsection may be omitted on a first reading.

$$\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n,$$

where each Γ_k , $k = 1, 2, \dots, n$, is a grid path in an open disc $D_k \subseteq \mathcal{R}$, joining two points in D_k . Figure 3.6 illustrates this construction, and shows the need for a finite sequence of overlapping discs D_1, D_2, \dots, D_n which link α to β .

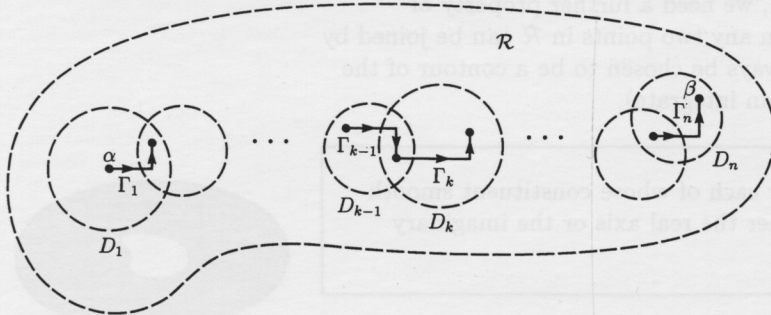


Figure 3.6

To show that such a sequence of discs always exists, we introduce the following definition and theorem. The proof of the Grid Path Theorem then follows almost immediately.

Definition A paving of a path $\Gamma: \gamma(t)$ ($t \in [a, b]$) is a finite sequence D_k , $k = 1, 2, \dots, n$, of open discs such that there are numbers t_k , $k = 0, 1, \dots, n$, satisfying

$$a = t_0 < t_1 < \dots < t_n = b$$

and

$$\gamma([t_{k-1}, t_k]) \subseteq D_k, \quad k = 1, 2, \dots, n.$$

We also say that the discs D_k , $k = 1, 2, \dots, n$, **pave** Γ .

For each k , the part of Γ defined by the restriction of γ to $[t_{k-1}, t_k]$ lies entirely in D_k .

We can now state the theorem, which is illustrated in Figure 3.7.

Theorem 3.7 Paving Theorem

Any path $\Gamma: \gamma(t)$ ($t \in [a, b]$) lying in a region \mathcal{R} can be paved by open discs D_k , $k = 1, 2, \dots, n$, such that

$$\bigcup_{k=1}^n D_k \subseteq \mathcal{R}.$$

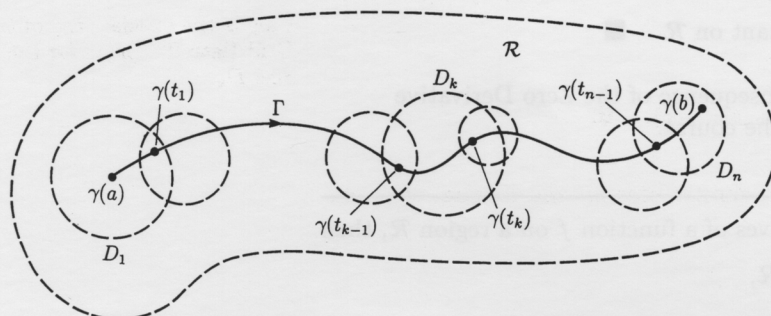


Figure 3.7 Paving a path

Proof Let $\alpha = \gamma(a)$ and $\beta = \gamma(b)$ be the initial and final points of Γ . Because \mathcal{R} is open, there is an open disc $\{z : |z - \alpha| < \varepsilon\}$, say, lying in \mathcal{R} . Since γ is continuous, there is a $\delta > 0$ such that

$$t \in [a, a + \delta] \implies |\gamma(t) - \gamma(a)| < \varepsilon.$$

Hence $\gamma([a, a + \frac{1}{2}\delta]) \subseteq \{z : |z - \alpha| < \varepsilon\}$ and so it is possible to start paving Γ with the disc $D_1 = \{z : |z - \alpha| < \varepsilon\}$. (See Figure 3.8.)

Now consider the set S of real numbers $s \in [a, b]$ such that the path determined by $\gamma(t)$ ($t \in [a, s]$) can be paved. We have just seen that $a + \frac{1}{2}\delta \in S$, and we want to show that $b \in S$. Clearly $S \subseteq [a, b]$ and it has the property that if $s \in S$ and $a \leq t < s$, then $t \in S$ also. Hence S is an interval with left-hand endpoint a and right-hand endpoint c , say, where $a < c \leq b$. We shall show that $c \in S$ and then that $c = b$, thus completing the proof.

Because \mathcal{R} is open, there is a disc $D' = \{z : |z - \gamma(c)| < \varepsilon'\}$, say, lying in \mathcal{R} . Since γ is continuous, there is a $\delta' > 0$ such that

$$\begin{aligned} t \in [a, b], |t - c| < \delta' &\implies |\gamma(t) - \gamma(c)| < \varepsilon' \\ &\implies \gamma(t) \in D'. \end{aligned} \quad (*)$$

Now, if $s \in [a, b]$ and $c - \delta' < s < c$, then $s \in S$ and so the path determined by $\gamma([a, s])$ can be paved. Since $\gamma(s) \in D'$, we can add D' to this paving, to deduce, by (*), that

$$\gamma([a, t]) \text{ can be paved, for } t \in [a, b] \text{ with } c \leq t < c + \delta'. \quad (\dagger)$$

It follows that $c \in S$ and, moreover, that $c = b$ (since otherwise (\dagger) would contradict the definition of c). Hence $b \in S$, as required. ■

We can now prove the Grid Path Theorem.

Theorem 3.5 Grid Path Theorem

If \mathcal{R} is a region, then any two points in \mathcal{R} can be joined by a grid path in \mathcal{R} .

Proof Let α and β be any two points in \mathcal{R} . Since \mathcal{R} is a region, α and β can be joined by a path $\Gamma : \gamma(t)$ ($t \in [a, b]$). Then $\alpha = \gamma(a)$, $\beta = \gamma(b)$.

By the Paving Theorem, the path Γ can be paved by open discs $D_k, k = 1, 2, \dots, n$, such that

$$\bigcup_{k=1}^n D_k \subseteq \mathcal{R};$$

that is, there are numbers t_0, t_1, \dots, t_n such that

$$a = t_0 < t_1 < \dots < t_n = b$$

and each path determined by $\gamma([t_{k-1}, t_k])$, $k = 1, 2, \dots, n$, lies in the open disc $D_k \subseteq \mathcal{R}$. Thus, for $k = 1, 2, \dots, n$, $\gamma(t_{k-1})$ can be joined to $\gamma(t_k)$ by a grid path Γ_k in D_k , and so

$$\Gamma_1 + \Gamma_2 + \dots + \Gamma_n$$

forms a grid path in \mathcal{R} joining $\alpha = \gamma(a)$ to $\beta = \gamma(b)$. ■

Remark We shall find a number of other similar uses for the Paving Theorem later in the course.

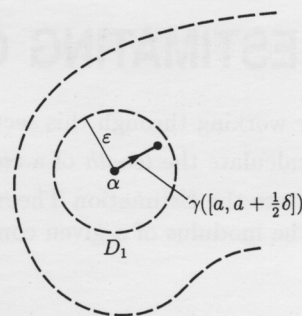


Figure 3.8

Indeed, if c is the least upper bound of S , then $S = [a, c]$ or $S = [a, c[$. The existence of such a least upper bound is justified in courses on real analysis.

This is the special case of the Grid Path Theorem for the open disc D_k .

4 ESTIMATING CONTOUR INTEGRALS

After working through this section, you should be able to:

- calculate the *length* of a smooth path or contour;
- state the Estimation Theorem and use it to obtain an upper estimate for the modulus of a given contour integral.

4.1 The length of a contour

It is easy to calculate the length of a line segment or a circular path, or of a contour made up from such paths. Sometimes, however, we need to find the lengths of other contours and, in order to do this, we first need to define exactly what we mean by the length of a general contour. Since any contour can be split into a finite number of smooth paths, it is enough to define the length of a smooth path.

The definition is given below, but first we give a heuristic argument which suggests the form of the definition.

Let $\Gamma : \gamma(t)$ ($t \in [a, b]$) be a smooth path, and let $L(\Gamma)$ be its length, which we wish to define. We now approximate Γ by a polygonal contour consisting of line segments joining the points

$$z_k = \gamma(t_k), \quad k = 0, 1, \dots, n,$$

where $a = t_0 < t_1 < \dots < t_n = b$ (see Figure 4.1). An approximation for $L(\Gamma)$ is then given by the sum of the lengths of these line segments

$$\sum_{k=1}^n |z_k - z_{k-1}| = \sum_{k=1}^n |\gamma(t_k) - \gamma(t_{k-1})|.$$

Now Γ is a smooth path, and so $\gamma'(t_k)$ exists for each $k = 1, 2, \dots, n$, and is approximately equal to $(\gamma(t_k) - \gamma(t_{k-1})) / (t_k - t_{k-1})$. Hence if n is large, then

$$|\gamma(t_k) - \gamma(t_{k-1})| \cong |\gamma'(t_k)| (t_k - t_{k-1}) \quad (\text{since } t_k > t_{k-1}),$$

so that

$$\begin{aligned} L(\Gamma) &\cong \sum_{k=1}^n |\gamma'(t_k)| (t_k - t_{k-1}) \\ &= \sum_{k=1}^n |\gamma'(t_k)| \delta t_k, \end{aligned}$$

where $\delta t_k = t_k - t_{k-1}$. Since this expression is a Riemann sum for the integral

$$\int_a^b |\gamma'(t)| dt,$$

we shall take this integral as our definition of $L(\Gamma)$.

Definitions Let $\Gamma : \gamma(t)$ ($t \in [a, b]$) be a smooth path. Then the **length of the path Γ** , denoted by $L(\Gamma)$, is

$$L(\Gamma) = \int_a^b |\gamma'(t)| dt.$$

The **length of a contour** is the sum of the lengths of its constituent smooth paths.

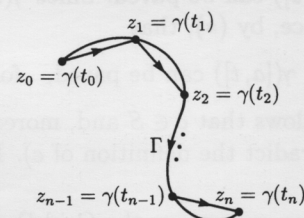


Figure 4.1 Approximating Γ

Since γ is smooth, γ' is continuous. Also the modulus function is continuous. Hence $t \mapsto |\gamma'(t)|$ is continuous and so it can be integrated.

It can be shown that the length of a contour is independent of the way that the contour is split into smooth paths.

The proof is similar to that for Theorem 2.1.

Remarks

- It is easy to check that the length of Γ is unchanged if γ is replaced by any equivalent parametrization.

2 It is also easy to check that

$$L(\tilde{\Gamma}) = L(\Gamma),$$

where $\tilde{\Gamma}$ is the reverse path of Γ .

Example 4.1

Use the above definition to verify that the length of the line segment joining the points 0 and $3 + 4i$ is 5 (see Figure 4.2).

Solution

The standard parametrization of the given line segment is

$$\gamma(t) = (3 + 4i)t \quad (t \in [0, 1]).$$

Then $\gamma'(t) = 3 + 4i$, so that

$$|\gamma'(t)| = \sqrt{3^2 + 4^2} = 5.$$

Thus the required length is

$$\int_0^1 5 \, dt = 5,$$

as expected. ■

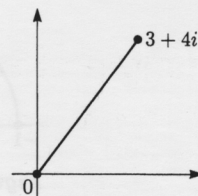


Figure 4.2

Problem 4.1

Use the above definition to find

- (a) the circumference of the circle with centre α and radius r ;
- (b) the length of the path with parametrization

$$\gamma(t) = t + i \cosh t \quad (t \in [0, 1]).$$

In this course we shall mainly use contours consisting of line segments and arcs of circles, whose lengths you knew before meeting the definition. However, the definition will be needed in theoretical work, and in examples where you do not already know the length of the path.

4.2 The Estimation Theorem

For a given function f which is continuous on a contour Γ , it may be impossible to evaluate $\int_{\Gamma} f(z) \, dz$ exactly. However, we can derive a result which gives an upper estimate for the *modulus* of the integral.

The result, which we state below, is of immense theoretical importance: indeed, many of the main proofs in complex analysis involve at least one use of it. When allied with the Residue Theorem, which you will meet in *Unit C1*, it also becomes an exceedingly useful technique in the evaluation of integrals.

We now state the Estimation Theorem; its proof will be given at the end of the section.

Theorem 4.1 Estimation Theorem

Let f be a function which is continuous on a contour Γ of length L , with

$$|f(z)| \leq M, \quad \text{for } z \in \Gamma.$$

Then

$$\left| \int_{\Gamma} f(z) \, dz \right| \leq ML.$$

Note that f is required to be continuous only on Γ .

The number M is an upper estimate for $|f|$ on Γ .

The following example illustrates the use of the Estimation Theorem.

Example 4.2

Find an upper estimate for

$$\left| \int_{\Gamma} \frac{e^z}{z} dz \right|,$$

where Γ is the upper half of the unit circle $\{z : |z| = 1\}$ shown in Figure 4.3.

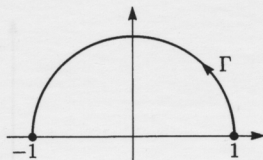


Figure 4.3

Solution

We have $L = \pi$, as Γ is a semi-circle of radius 1. To find an upper estimate for $|f|$ on Γ , where $f(z) = e^z/z$, note that

$$\begin{aligned} |e^z| &= e^{\operatorname{Re} z} \\ &\leq e, \quad \text{for } z \in \Gamma \quad (\text{since } \operatorname{Re} z \leq 1, \text{ for } |z| = 1), \end{aligned}$$

and

$$|z| = 1, \quad \text{for } z \in \Gamma.$$

Thus

$$|f(z)| = \left| \frac{e^z}{z} \right| = \frac{|e^z|}{|z|} \leq e, \quad \text{for } z \in \Gamma,$$

so we take $M = e$. Since $f(z) = e^z/z$ is continuous on $\mathbb{C} - \{0\}$ (by the Quotient Rule) and hence on Γ , it follows from the Estimation Theorem that

$$\left| \int_{\Gamma} \frac{e^z}{z} dz \right| \leq \pi e. \quad \blacksquare$$

Problem 4.2

Use the Estimation Theorem to find an upper estimate for

$$\left| \int_{\Gamma} \frac{e^{3z}}{z^2} dz \right|,$$

where Γ is

- (a) the circle $\{z : |z| = 5\}$;
- (b) the square contour with vertices at

$$5 + 5i, -5 + 5i, -5 - 5i, 5 - 5i$$

(traversed once anticlockwise and having initial point $5 + 5i$).

When determining M , we often need to use the Triangle Inequality. Recall that the Triangle Inequality can be stated in the following forms.

Usual form: if $z_1, z_2 \in \mathbb{C}$, then

$$|z_1 + z_2| \leq |z_1| + |z_2|; \quad |z_1 - z_2| \leq |z_1| + |z_2|.$$

Backwards form: if $z_1, z_2 \in \mathbb{C}$, then

$$|z_1 + z_2| \geq ||z_1| - |z_2||; \quad |z_1 - z_2| \geq ||z_1| - |z_2||.$$

The following examples illustrate the use of the Triangle Inequality in the estimation of integrals.

The identity $|e^z| = e^{\operatorname{Re} z}$ was established in Unit A2, Theorem 4.1(b).

Unit A1, Theorem 5.1 and its corollary

Example 4.3

Find an upper estimate for

$$\left| \int_{\Gamma} \frac{1}{z^2 + 1} dz \right|,$$

where Γ is the upper half of the circle $\{z : |z| = 3\}$.

Solution

We have $L = 3\pi$, as Γ is a semi-circle of radius 3. To find a value for M note that, by the Triangle Inequality (backwards form), we have

$$|z^2 + 1| \geq |3^2 - 1| = 8, \quad \text{for } z \in \Gamma.$$

Thus

$$\left| \frac{1}{z^2 + 1} \right| \leq \frac{1}{8}, \quad \text{for } z \in \Gamma,$$

so we take $M = \frac{1}{8}$. Since $f(z) = 1/(z^2 + 1)$ is continuous on $\mathbb{C} - \{i, -i\}$ and hence on Γ , it follows from the Estimation Theorem that

$$\left| \int_{\Gamma} \frac{1}{z^2 + 1} dz \right| \leq \frac{3\pi}{8}. \quad \blacksquare$$

Note that, to get an upper estimate for $|1/(z^2 + 1)|$, we need a *lower* estimate for $|z^2 + 1|$.

Example 4.4

Find an upper estimate for

$$\left| \int_{\Gamma} \frac{z^2 - 4z - 3}{(z^2 - 7)(z^2 + 2)} dz \right|,$$

where Γ is the circle $\{z : |z| = 2\}$.

Solution

We have $L = 4\pi$, as Γ is a circle of radius 2. To find a value for M note that, by the Triangle Inequality (usual form), we have

$$\begin{aligned} |z^2 - 4z - 3| &\leq |z^2| + |-4z| + |-3| \\ &= 4 + 8 + 3 = 15, \quad \text{for } z \in \Gamma, \end{aligned}$$

and, by the Triangle Inequality (backwards form), we have

$$|z^2 - 7| \geq |2^2 - 7| = 3, \quad \text{for } z \in \Gamma,$$

and

$$|z^2 + 2| \geq |2^2 - 2| = 2, \quad \text{for } z \in \Gamma.$$

Thus

$$\left| \frac{z^2 - 4z - 3}{(z^2 - 7)(z^2 + 2)} \right| \leq \frac{15}{3 \times 2} = \frac{5}{2}, \quad \text{for } z \in \Gamma,$$

so we take $M = \frac{5}{2}$. Since $f(z) = (z^2 - 4z - 3)/((z^2 - 7)(z^2 + 2))$ is continuous on $\mathbb{C} - \{\sqrt{7}, -\sqrt{7}, i\sqrt{2}, -i\sqrt{2}\}$ and hence on Γ , it follows from the Estimation Theorem that

$$\left| \int_{\Gamma} \frac{z^2 - 4z - 3}{(z^2 - 7)(z^2 + 2)} dz \right| \leq \frac{5}{2} \times 4\pi = 10\pi. \quad \blacksquare$$

Here, we need to find an *upper* estimate for $|z^2 - 4z - 3|$, and *lower* estimates for $|z^2 - 7|$ and $|z^2 + 2|$.

Problem 4.3

Use the Estimation Theorem to find an upper estimate for

$$\left| \int_{\Gamma} \frac{3z - 4}{2z - 5} dz \right|,$$

where Γ is the circle $\{z : |z| = 3\}$.

Our final example involves both exponential functions and the use of the Triangle Inequality.

Example 4.5

Find an upper estimate for

$$\left| \int_{\Gamma} \frac{(z - 3i)e^{iz}}{z^2 + 4} dz \right|,$$

where Γ is the upper half of the circle $\{z : |z| = 5\}$.

Solution

We have $L = 5\pi$, as Γ is a semi-circle of radius 5. To find a value for M note that, by the Triangle Inequality (both forms), we have

$$|z - 3i| \leq |z| + |-3i| = 5 + 3 = 8, \quad \text{for } z \in \Gamma,$$

and

$$|z^2 + 4| \geq |5^2 - 4| = 21, \quad \text{for } z \in \Gamma.$$

For the exponential term, we can write

$$\begin{aligned} |e^{iz}| &= |e^{i(x+iy)}| \\ &= |e^{ix}| |e^{-y}| \\ &= e^{-y} \\ &\leq e^0 = 1, \quad \text{for } z \in \Gamma \quad (\text{since } y \geq 0, \text{ for } z \in \Gamma). \end{aligned}$$

Thus

$$\left| \frac{(z - 3i)e^{iz}}{z^2 + 4} \right| \leq \frac{8 \times 1}{21} = \frac{8}{21}, \quad \text{for } z \in \Gamma,$$

so we take $M = \frac{8}{21}$. Since $f(z) = (z - 3i)e^{iz}/(z^2 + 4)$ is continuous on $\mathbb{C} - \{2i, -2i\}$ (by the Composition Rule and the Combination Rules) and hence on Γ , it follows from the Estimation Theorem that

$$\left| \int_{\Gamma} \frac{(z - 3i)e^{iz}}{z^2 + 4} dz \right| \leq \frac{8}{21} \times 5\pi = \frac{40}{21}\pi. \quad \blacksquare$$

Alternatively,

$$|e^{iz}| = e^{\operatorname{Re}(iz)} = e^{-y}.$$

Note that the inequality

$$|e^{iz}| \leq e^{|iz|} = e^{|z|}$$

gives a much poorer estimate here.

Problem 4.4

Use the Estimation Theorem to find an upper estimate for

$$\left| \int_{\Gamma} \frac{e^{2iz}}{z^2 - 9} dz \right|,$$

where Γ is the upper half of the circle $\{z : |z| = 4\}$.

4.3 Proof of the Estimation Theorem

We first present a useful lemma which extends the Triangle Inequality for integrals of a real variable (Section 1, Frame 6) to complex-valued functions.

This subsection may be omitted on a first reading.

Lemma 4.1 Let $g : [a, b] \rightarrow \mathbb{C}$ be a continuous function. Then

$$\left| \int_a^b g(t) dt \right| \leq \int_a^b |g(t)| dt.$$

Proof We first write the complex number $\int_a^b g(t) dt$ in polar form, as $\int_a^b g(t) dt = re^{i\theta}$. Then

$$\begin{aligned} r &= e^{-i\theta} \int_a^b g(t) dt = \int_a^b e^{-i\theta} g(t) dt \quad (\text{since } e^{-i\theta} \text{ is a constant}) \\ &= \int_a^b \operatorname{Re}(e^{-i\theta} g(t)) dt + i \int_a^b \operatorname{Im}(e^{-i\theta} g(t)) dt. \end{aligned}$$

Since r is real, the second integral on the right must be 0; that is,

$$r = \int_a^b \operatorname{Re}(e^{-i\theta} g(t)) dt.$$

But $\left| \int_a^b g(t) dt \right| = |re^{i\theta}| = r$, since $|e^{i\theta}| = 1$, so that

$$\begin{aligned} \left| \int_a^b g(t) dt \right| &= \int_a^b \operatorname{Re}(e^{-i\theta} g(t)) dt \\ &\leq \int_a^b |e^{-i\theta} g(t)| dt \quad (\text{since } \operatorname{Re} z \leq |z|) \\ &= \int_a^b |g(t)| dt \quad (\text{since } |e^{-i\theta}| = 1). \quad \blacksquare \end{aligned}$$

Note that if $\int_a^b g(t) dt = 0$ then the result is evident.

Here, we use the Monotonicity Inequality for real integrals (Frame 6).

We can now prove the Estimation Theorem, which we repeat for convenience.

Theorem 4.1 Estimation Theorem

Let f be a function which is continuous on a contour Γ of length L , with

$$|f(z)| \leq M, \quad \text{for } z \in \Gamma.$$

Then

$$\left| \int_{\Gamma} f(z) dz \right| \leq ML.$$

Proof As usual, the proof is in two parts. We first prove the result in the case when Γ is a smooth path, and then extend the proof to contours.

(a) Let $\Gamma: \gamma(t)$ ($t \in [a, b]$) be a smooth path. Then

$$\int_{\Gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt.$$

Applying Lemma 4.1 with $g(t) = f(\gamma(t)) \gamma'(t)$, we obtain

$$\begin{aligned} \left| \int_{\Gamma} f(z) dz \right| &\leq \int_a^b |f(\gamma(t)) \gamma'(t)| dt \\ &= \int_a^b |f(\gamma(t))| |\gamma'(t)| dt \\ &\leq \int_a^b M |\gamma'(t)| dt \quad (\text{since } |f(z)| \leq M, \text{ for } z \in \Gamma) \\ &= M \int_a^b |\gamma'(t)| dt \\ &= ML \quad (\text{by the definition of } L). \end{aligned}$$

This proves the result in this case.

(b) To extend the proof to a general contour Γ , we argue as follows.

Let $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$, and let $L_k = L(\Gamma_k)$, for $k = 1, 2, \dots, n$, so that Γ has length $L = L_1 + L_2 + \cdots + L_n$. Then

$$\int_{\Gamma} f(z) dz = \int_{\Gamma_1} f(z) dz + \int_{\Gamma_2} f(z) dz + \cdots + \int_{\Gamma_n} f(z) dz,$$

and so, by the extended form of the Triangle Inequality, we have

$$\begin{aligned} \left| \int_{\Gamma} f \right| &\leq \left| \int_{\Gamma_1} f \right| + \left| \int_{\Gamma_2} f \right| + \cdots + \left| \int_{\Gamma_n} f \right| \\ &\leq ML_1 + ML_2 + \cdots + ML_n \quad (\text{by part (a)}) \\ &= M(L_1 + L_2 + \cdots + L_n) \\ &= ML. \end{aligned}$$

This completes the proof. ■

EXERCISES

Section 1

There are no exercises for this section.

Section 2

Exercise 2.1 Evaluate each of the following integrals (using the standard parametrization for the path Γ in each case).

(a) (i) $\int_{\Gamma} z dz$ (ii) $\int_{\Gamma} \operatorname{Im} z dz$ (iii) $\int_{\Gamma} \bar{z} dz$,

where Γ is the line segment from 1 to i .

(b) (i) $\int_{\Gamma} \bar{z} dz$ (ii) $\int_{\Gamma} z^2 dz$,

where Γ is the unit circle $\{z : |z| = 1\}$.

(c) (i) $\int_{\Gamma} \frac{1}{z} dz$ (ii) $\int_{\Gamma} |z| dz$,

where Γ is the upper half of the circle with centre 0 and radius 2 from 2 to -2 .

Exercise 2.2 Which of the following pairs of smooth paths are equivalent?

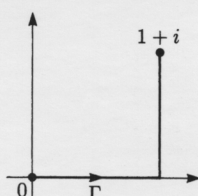
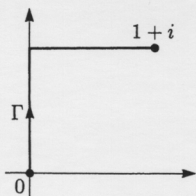
(a) $\Gamma_1 : \gamma_1(t) = 1 + i(1 + 2t)$ ($t \in [0, 1]$); $\Gamma_2 : \gamma_2(t) = 1 + i(1 - 2t)$ ($t \in [0, 1]$).

(b) $\Gamma_1 : \gamma_1(t) = 1 + e^{it}$ ($t \in [0, \pi]$); $\Gamma_2 : \gamma_2(t) = 1 + e^{2it}$ ($t \in [0, \frac{1}{2}\pi]$).

Exercise 2.3 Evaluate $\int_{\Gamma} \operatorname{Re} z dz$, where

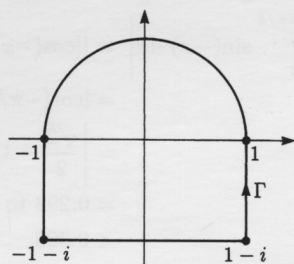
(a) Γ is the contour from 0 to $1 + i$ shown in the left-hand figure;

(b) Γ is the contour from 0 to $1 + i$ shown in the right-hand figure.



Exercise 2.4

- (a) The contour Γ shown below consists of three line segments and a semi-circle.



- (i) Split Γ into four constituent smooth paths and write down the standard (smooth) parametrization for each of these paths. (Take $1 - i$ as the initial and final point of Γ .)
- (ii) Write down the corresponding smooth parametrization for each of the constituent paths of $\tilde{\Gamma}$, the reverse contour of Γ .
- (b) Evaluate (i) $\int_{\Gamma} \bar{z} dz$ and (ii) $\int_{\tilde{\Gamma}} \bar{z} dz$, where Γ is as defined in part (a).

Section 3

Exercise 3.1 Use the results of Section 3 to evaluate $\int_{\Gamma} f(z) dz$, where Γ is any contour from $-i$ to i , for each of the following functions f .

- (a) $f(z) = 1$ (b) $f(z) = z$ (c) $f(z) = 5z^4 + 3iz^2$
- (d) $f(z) = (1 + 2iz)^9$ (e) $f(z) = e^{-iz}$ (f) $f(z) = \sin z$
- (g) $f(z) = ze^{z^2}$ (h) $f(z) = z^3 \cosh(z^4)$ (i) $f(z) = ze^z$

Exercise 3.2 Use the results of Section 3 to evaluate each of the following integrals. (In each case pay special attention to the hypotheses of the theorems you use.)

- (a) $\int_{\Gamma} \frac{1}{z} dz$, where Γ is the arc of the circle $\{z : |z| = 1\}$ from $-i$ to i passing through 1.
- (b) $\int_{\Gamma} \sqrt{z} dz$, where Γ is as in part (a).
- (c) $\int_{\Gamma} \sin^2 z dz$, where Γ is the unit circle $\{z : |z| = 1\}$.
- (d) $\int_{\Gamma} \frac{1}{z^3} dz$, where Γ is the circle $\{z : |z| = 27\}$.

Exercise 3.3 Construct a grid path from α to β in the domain of the function \tan , for each of the following cases.

- (a) $\alpha = 1, \beta = 6$ (b) $\alpha = \frac{1}{2}\pi + 2i, \beta = -\frac{3}{2}\pi - i$

Section 4

Exercise 4.1 Find an upper estimate for $\left| \int_{\Gamma} f(z) dz \right|$, for each of the following functions, where Γ is the circle $\{z : |z| = 3\}$.

- (a) $f(z) = z + 2$ (b) $f(z) = \frac{1}{z + 2}$ (c) $f(z) = \frac{z - 3}{z + 2}$
- (d) $f(z) = \frac{z^2 + 4}{z^2 - 4}$ (e) $f(z) = e^{z-3}$ (f) $f(z) = \frac{\sin z}{1 + z^2}$

SOLUTIONS TO THE PROBLEMS

Section 1

1.1 With $f(x) = x^3$ and

$$P_n = \{[0, 1/n], [1/n, 2/n], \dots, [(n-1)/n, 1]\},$$

we have

$$R(f, P_n) = \sum_{k=1}^n f(x_k) \delta x_k,$$

where

$$x_k = k/n, \quad k = 0, 1, 2, \dots, n,$$

and

$$\delta x_k = 1/n, \quad k = 1, 2, \dots, n.$$

Thus

$$\begin{aligned} R(f, P_n) &= \sum_{k=1}^n \left(\frac{k}{n}\right)^3 \cdot \frac{1}{n} \\ &= \frac{1}{n^4} (1^3 + 2^3 + \dots + n^3) \\ &= \frac{n^2(n+1)^2}{4n^4} \\ &= \frac{1}{4} (1 + 1/n)^2. \end{aligned}$$

$$\text{Hence } \lim_{n \rightarrow \infty} R(f, P_n) = \frac{1}{4}.$$

1.2 Since

$$e^{-x} \leq e^{-x^2} \leq \frac{1}{1+x^2}, \quad \text{for } 0 \leq x \leq 1,$$

it follows from the Monotonicity Inequality that

$$\int_0^1 e^{-x} dx \leq \int_0^1 e^{-x^2} dx \leq \int_0^1 \frac{1}{1+x^2} dx.$$

Hence

$$[-e^{-x}]_0^1 \leq \int_0^1 e^{-x^2} dx \leq [\tan^{-1} x]_0^1;$$

that is,

$$1 - e^{-1} \leq \int_0^1 e^{-x^2} dx \leq \frac{1}{4}\pi,$$

or

$$0.63 \leq \int_0^1 e^{-x^2} dx \leq 0.79.$$

(In fact, $\int_0^1 e^{-x^2} dx = 0.75$ to 2 decimal places.)

1.3 We have

$$\begin{aligned} \left| \int_0^{\pi/4} \sin(-x) dx \right| &\leq \int_0^{\pi/4} |\sin(-x)| dx \\ &\quad (\text{Triangle Inequality}) \\ &= \int_0^{\pi/4} \sin x dx \quad (|\sin(-x)| = \sin x) \\ &\leq \int_0^{\pi/4} x dx \quad (\sin x \leq x, \text{ for } x \geq 0) \\ &= \left[\frac{1}{2}x^2\right]_0^{\pi/4} \\ &= \frac{1}{32}\pi^2 = 0.308 \text{ to 3 decimal places.} \end{aligned}$$

Evaluating the integral, we obtain

$$\begin{aligned} \left| \int_0^{\pi/4} \sin(-x) dx \right| &= |[\cos(-x)]_0^{\pi/4}| \\ &= |\cos(-\pi/4) - \cos 0| \\ &= \left| \frac{\sqrt{2}}{2} - 1 \right| \\ &= 0.293 \text{ to 3 decimal places} \\ &< 0.308. \end{aligned}$$

Section 2

2.1 (a) Here $\gamma(t) = 2t(1+i)$ ($t \in [0, \frac{1}{2}]$). Let $f(z) = \bar{z}$.

Then

$$f(\gamma(t)) = \overline{2t(1+i)} = 2t(1-i)$$

and, since $\gamma'(t) = 2(1+i)$, we obtain

$$\begin{aligned} \int_{\gamma} \bar{z} dz &= \int_0^{1/2} 2t(1-i) \cdot 2(1+i) dt \\ &= \int_0^{1/2} 8t dt + i \int_0^{1/2} 0 dt \\ &= [4t^2]_0^{1/2} = 1, \text{ as in Example 2.1.} \end{aligned}$$

(b) Here $\gamma(t) = e^{3it}$ ($t \in [0, \frac{2}{3}\pi]$). Then

$$z = e^{3it}, \quad 1/z = e^{-3it} \quad \text{and} \quad dz = 3ie^{3it} dt.$$

Hence

$$\begin{aligned} \int_{\gamma} \frac{1}{z} dz &= \int_0^{2\pi/3} e^{-3it} \cdot 3ie^{3it} dt \\ &= i \int_0^{2\pi/3} 3 dt \\ &= i[3t]_0^{2\pi/3} = 2\pi i, \text{ as in Example 2.2.} \end{aligned}$$

(Of course, we could have employed the z, dz formulation in part (a) as well.)

2.2 (a) The paths

$$\Gamma_1 : \gamma_1(t) = 2 + i(t-1) \quad (t \in [1, 2])$$

and

$$\Gamma_2 : \gamma_2(t) = 2 + it \quad (t \in [0, 1])$$

are equivalent because $\gamma_1(t) = \gamma_2(h(t))$, where

$$h(t) = t-1 \quad (t \in [1, 2]).$$

(The function h is such that: $h(1) = 0$, $h(2) = 1$; $h'(t) = 1$, which is continuous and positive on $[1, 2]$.)

(b) The paths

$$\Gamma_1 : \gamma_1(t) = (1-t)i + t \quad (t \in [0, 1])$$

and

$$\Gamma_2 : \gamma_2(t) = (1-t) + ti \quad (t \in [0, 1])$$

are not equivalent. For, any function $h : [0, 1] \rightarrow [0, 1]$ satisfying $\gamma_1(t) = \gamma_2(h(t))$ is such that

$$(1-t)i + t = (1-h(t)) + h(t)i.$$

Equating real and imaginary parts, we obtain

$$h(t) = 1-t;$$

but $h'(t) = -1$, which is not positive.

(In fact, Γ_2 is the 'reverse path' of Γ_1 : see Subsection 2.3.)

2.3 (a) The standard parametrization of Γ , the line segment from 0 to $1 + 2i$, is

$$\gamma(t) = (1 + 2i)t \quad (t \in [0, 1]).$$

Then

$$z = (1 + 2i)t, \quad \operatorname{Re} z = t, \quad dz = (1 + 2i) dt.$$

Hence

$$\begin{aligned} \int_{\Gamma} \operatorname{Re} z \, dz &= \int_0^1 t \cdot (1 + 2i) dt \\ &= \int_0^1 t \, dt + i \int_0^1 2t \, dt \\ &= \left[\frac{1}{2} t^2 \right]_0^1 + i \left[t^2 \right]_0^1 \\ &= \frac{1}{2} + i. \end{aligned}$$

(b) The standard parametrization of Γ , the circle with centre α and radius r , is

$$\gamma(t) = \alpha + re^{it} \quad (t \in [0, 2\pi]).$$

Then

$$z = \alpha + re^{it}, \quad 1/(z - \alpha)^2 = 1/(r^2 e^{2it}), \quad dz = rie^{it} dt.$$

Hence

$$\begin{aligned} \int_{\Gamma} \frac{1}{(z - \alpha)^2} dz &= \int_0^{2\pi} \frac{rie^{it}}{r^2 e^{2it}} dt \\ &= \int_0^{2\pi} \frac{1}{r} ie^{-it} dt \\ &= \int_0^{2\pi} \frac{1}{r} i (\cos t - i \sin t) dt \\ &= \int_0^{2\pi} \frac{1}{r} \sin t \, dt + i \int_0^{2\pi} \frac{1}{r} \cos t \, dt \\ &= \left[-\frac{1}{r} \cos t \right]_0^{2\pi} + i \left[\frac{1}{r} \sin t \right]_0^{2\pi} \\ &= 0 + 0i = 0. \end{aligned}$$

2.4 (a) $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$, where Γ_1 is the line segment from 0 to 1, Γ_2 is the line segment from 1 to $1 + i$ and Γ_3 is the line segment from $1 + i$ to i . We choose to use the associated standard parametrizations

$$\gamma_1(t) = t \quad (t \in [0, 1]),$$

$$\gamma_2(t) = 1 + it \quad (t \in [0, 1]),$$

$$\gamma_3(t) = 1 - t + i \quad (t \in [0, 1]).$$

Then $\gamma_1'(t) = 1$, $\gamma_2'(t) = i$, $\gamma_3'(t) = -1$. Hence

$$\begin{aligned} \int_{\Gamma} \bar{z} \, dz &= \int_{\Gamma_1} \bar{z} \, dz + \int_{\Gamma_2} \bar{z} \, dz + \int_{\Gamma_3} \bar{z} \, dz \\ &= \int_0^1 t \cdot 1 \, dt + \int_0^1 (1 - it) \cdot i \, dt \\ &\quad + \int_0^1 (1 - t - i) \cdot (-1) \, dt \\ &= \int_0^1 t \, dt + \left(\int_0^1 t \, dt + i \int_0^1 1 \, dt \right) \\ &\quad + \left(\int_0^1 (t - 1) \, dt + i \int_0^1 1 \, dt \right) \\ &= \left[\frac{1}{2} t^2 \right]_0^1 + \left(\left[\frac{1}{2} t^2 \right]_0^1 + i \left[t \right]_0^1 \right) \\ &\quad + \left(\left[\frac{1}{2} t^2 - t \right]_0^1 + i \left[t \right]_0^1 \right) \\ &= \frac{1}{2} + \left(\frac{1}{2} + i \right) + \left(\frac{1}{2} - 1 + i \right) = \frac{1}{2} + 2i. \end{aligned}$$

(b) $\Gamma = \Gamma_1 + \Gamma_2$, where Γ_1 is the line segment from -1 to 1 and Γ_2 is the upper half of the circle with centre 0 from 1 to -1 . We choose to use the following parametrizations

$$\gamma_1(t) = t \quad (t \in [-1, 1]),$$

$$\gamma_2(t) = e^{it} \quad (t \in [0, \pi]).$$

Then $\gamma_1'(t) = 1$, $\gamma_2'(t) = ie^{it}$. Hence

$$\begin{aligned} \int_{\Gamma} \bar{z} \, dz &= \int_{\Gamma_1} \bar{z} \, dz + \int_{\Gamma_2} \bar{z} \, dz \\ &= \int_{-1}^1 t \cdot 1 \, dt + \int_0^{\pi} e^{-it} \cdot ie^{it} \, dt \\ &= \int_{-1}^1 t \, dt + i \int_0^{\pi} 1 \, dt \\ &= \left[\frac{1}{2} t^2 \right]_{-1}^1 + i \left[t \right]_0^{\pi} \\ &= 0 + i\pi = \pi i. \end{aligned}$$

2.5 Since $a = 0$ and $b = 2$, the reverse path is $\tilde{\Gamma} : \tilde{\gamma}$ where

$$\tilde{\gamma}(t) = \gamma(2 - t)$$

$$= 2 + i - (2 - t)$$

$$= t + i \quad (t \in [0, 2]).$$

2.6 (a) In Example 2.2, we used the parametrization

$$\gamma(t) = e^{it} \quad (t \in [0, 2\pi]).$$

For the reverse path $\tilde{\Gamma}$ we use the parametrization

$$\tilde{\gamma}(t) = \gamma(2\pi - t) = e^{i(2\pi - t)} \quad (t \in [0, 2\pi]).$$

Since $e^{2\pi i} = 1$, we have

$$\tilde{\gamma}(t) = e^{-it} \quad (t \in [0, 2\pi]),$$

and hence

$$\begin{aligned} \int_{\tilde{\Gamma}} \frac{1}{z} \, dz &= \int_0^{2\pi} \frac{1}{e^{-it}} \cdot (-ie^{-it}) \, dt \\ &= -i \int_0^{2\pi} 1 \, dt \\ &= -2\pi i \quad \left(= - \int_{\Gamma} \frac{1}{z} \, dz \right). \end{aligned}$$

(b) In Example 2.3, we wrote $\Gamma = \Gamma_1 + \Gamma_2$ where Γ_1 is the line segment from 0 to 1, with parametrization

$$\gamma_1(t) = t \quad (t \in [0, 1]),$$

and Γ_2 is the line segment from 1 to $1 + i$, with parametrization

$$\gamma_2(t) = 1 + it \quad (t \in [0, 1]).$$

For the reverse contour $\tilde{\Gamma} = \tilde{\Gamma}_2 + \tilde{\Gamma}_1$, we use the parametrizations

$$\tilde{\gamma}_1(t) = \gamma_1(1 - t) = 1 - t \quad (t \in [0, 1])$$

and

$$\tilde{\gamma}_2(t) = \gamma_2(1 - t) = 1 + i(1 - t) \quad (t \in [0, 1]).$$

Hence

$$\begin{aligned}
 \int_{\tilde{\Gamma}} z^2 dz &= \int_{\tilde{\Gamma}_2} z^2 dz + \int_{\tilde{\Gamma}_1} z^2 dz \\
 &= \int_0^1 (1 + i(1-t))^2 \cdot (-i) dt \\
 &\quad + \int_0^1 (1-t)^2 \cdot (-1) dt \\
 &= \left(\int_0^1 2(1-t) dt + i \int_0^1 (t^2 - 2t) dt \right) \\
 &\quad - \int_0^1 (1-t)^2 dt \\
 &= [2t - t^2]_0^1 + i \left[\frac{1}{3}t^3 - t^2 \right]_0^1 + \left[\frac{1}{3}(1-t)^3 \right]_0^1 \\
 &= 1 + i \left(-\frac{2}{3} \right) - \frac{1}{3} \\
 &= \frac{2}{3} - \frac{2}{3}i \quad \left(= - \int_{\Gamma} z^2 dz \right).
 \end{aligned}$$

Section 3

3.1 (a) $F(z) = \frac{1}{3i} e^{3iz} \quad (z \in \mathbb{C})$

(b) $F(z) = i(1+iz)^{-1} = (z-i)^{-1} \quad (z \in \mathbb{C} - \{i\})$

(c) $F(z) = \text{Log } z \quad (\text{Re } z > 0)$

3.2 Let $f(z) = e^{3iz}$, $F(z) = \frac{1}{3i} e^{3iz}$, and $\mathcal{R} = \mathbb{C}$. Then f is continuous on \mathcal{R} , and F is a primitive of f on \mathcal{R} . Thus, by the Fundamental Theorem of Calculus,

$$\begin{aligned}
 \int_{\Gamma} e^{3iz} dz &= F(-2) - F(2) \\
 &= \frac{1}{3i} (e^{-6i} - e^{6i}) = -\frac{2}{3} \sin 6.
 \end{aligned}$$

(Note that such exponential expressions should be simplified.)

3.3 (a) Let $f(z) = e^{-\pi z}$, $F(z) = -e^{-\pi z}/\pi$, and $\mathcal{R} = \mathbb{C}$. Then f is continuous on \mathcal{R} , and F is a primitive of f on \mathcal{R} . Thus, by the Fundamental Theorem of Calculus,

$$\begin{aligned}
 \int_{\Gamma} e^{-\pi z} dz &= F(i) - F(-i) \\
 &= (-e^{-\pi i}/\pi) - (-e^{\pi i}/\pi) = 0.
 \end{aligned}$$

(b) Let $f(z) = (3z-1)^4$, $F(z) = \frac{1}{15} (3z-1)^5$, and $\mathcal{R} = \mathbb{C}$. Then f is continuous on \mathcal{R} , and F is a primitive of f on \mathcal{R} . Thus, by the Fundamental Theorem of Calculus,

$$\begin{aligned}
 \int_{\Gamma} (3z-1)^4 dz &= F(2i + \frac{1}{3}) - F(2) \\
 &= \frac{1}{15} (6i)^5 - \frac{1}{15} (5)^5 \\
 &= \frac{1}{15} (7776i - 3125).
 \end{aligned}$$

(c) Let $f(z) = \sinh z$, $F(z) = \cosh z$, and $\mathcal{R} = \mathbb{C}$. Then f is continuous on \mathcal{R} , and F is a primitive of f on \mathcal{R} . Thus, by the Fundamental Theorem of Calculus,

$$\begin{aligned}
 \int_{\Gamma} \sinh z dz &= F(1) - F(i) \\
 &= \cosh 1 - \cosh i \\
 &= \cosh 1 - \cos 1.
 \end{aligned}$$

(d) The integrand $e^{\sin z} \cos z$ can be written as $\exp(\sin z) \cdot \sin' z$ which equals $(\exp \circ \sin)'(z)$.

So let $f(z) = \exp(\sin z) \cos z$, $F(z) = \exp(\sin z)$, and $\mathcal{R} = \mathbb{C}$. Then f is continuous on \mathcal{R} , and F is a primitive of f on \mathcal{R} . Thus, by the Fundamental Theorem of Calculus,

$$\begin{aligned}
 \int_{\Gamma} e^{\sin z} \cos z dz &= F(\tfrac{1}{2}\pi) - F(0) \\
 &= \exp(\sin \tfrac{1}{2}\pi) - \exp(\sin 0) \\
 &= e - 1.
 \end{aligned}$$

(e) The integrand $\sin z / \cos^2 z$ can be written as

$$-\frac{1}{\cos^2 z} \cos' z,$$

which equals

$$(h \circ \cos)'(z), \quad \text{where } h(z) = 1/z.$$

So, let $f(z) = \sin z / \cos^2 z$, $F(z) = h(\cos z) = 1/\cos z$, $\mathcal{R} = \mathbb{C} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$. Then f is continuous on \mathcal{R} , and F is a primitive of f on \mathcal{R} . Thus, by the Fundamental Theorem of Calculus,

$$\begin{aligned}
 \int_{\Gamma} \frac{\sin z}{\cos^2 z} dz &= F(\pi) - F(0) \\
 &= \frac{1}{\cos \pi} - \frac{1}{\cos 0} \\
 &= -1 - 1 = -2.
 \end{aligned}$$

(In this solution, note that the region \mathcal{R} does not contain the point $\frac{1}{2}\pi$, as $\cos \frac{1}{2}\pi = 0$; thus Γ cannot be chosen to be a path which contains $\frac{1}{2}\pi$. In particular, the real

integral $\int_0^{\pi} \frac{\sin x}{\cos^2 x} dx$ does not exist.)

3.4 (a) We take $f(z) = z$, $g(z) = \sinh z$, and $\mathcal{R} = \mathbb{C}$. Then f and g are analytic on \mathcal{R} , and $f'(z) = 1$ and $g'(z) = \cosh z$ are continuous on \mathcal{R} .

Integrating by parts, we obtain

$$\begin{aligned}
 \int_{\Gamma} z \cosh z dz &= [z \sinh z]_0^{\pi i} - \int_{\Gamma} 1 \cdot \sinh z dz \\
 &= (\pi i \sinh \pi i - 0) - [\cosh z]_0^{\pi i} \\
 &= 0 - (-1 - 1) = 2.
 \end{aligned}$$

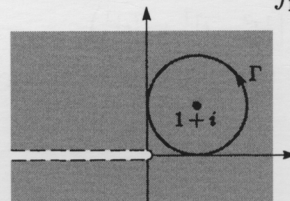
(b) We take $f(z) = \text{Log } z$, $g(z) = z$ and $\mathcal{R} = \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$. Then f and g are analytic on \mathcal{R} , and $f'(z) = 1/z$ and $g'(z) = 1$ are continuous on \mathcal{R} .

Integrating by parts, we obtain

$$\begin{aligned}
 \int_{\Gamma} \text{Log } z dz &= [z \text{Log } z]_1^i - \int_{\Gamma} \frac{1}{z} \cdot z dz \\
 &= i \text{Log } i - \text{Log } 1 - [z]_1^i \\
 &= -\frac{1}{2}\pi - (i - 1) = (1 - \frac{1}{2}\pi) - i.
 \end{aligned}$$

3.5 (a) Let $f(z) = 1/z$, $F(z) = \text{Log } z$, and $\mathcal{R} = \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$. Then f is continuous on \mathcal{R} , F is a primitive of f on \mathcal{R} , and Γ lies in \mathcal{R} (see the figure).

Thus, by the Closed Contour Theorem, $\int_{\Gamma} 1/z dz = 0$.



(b) Let $f(z) = 1/z^2$, $F(z) = -1/z$ and $\mathcal{R} = \mathbb{C} - \{0\}$. Then f is continuous on \mathcal{R} , F is a primitive of f on \mathcal{R} , and Γ lies in \mathcal{R} . Thus, by the Closed Contour Theorem, $\int_{\Gamma} 1/z^2 dz = 0$.

3.6 Let $F(z) = F_1(z) - F_2(z)$.

Since F_1 and F_2 are both primitives of f on \mathcal{R} , F_1 and F_2 are analytic on \mathcal{R} , and

$$F_1'(z) = F_2'(z) = f(z), \quad \text{for all } z \text{ in } \mathcal{R};$$

thus, F is analytic on \mathcal{R} , and

$$F'(z) = F_1'(z) - F_2'(z) = 0, \quad \text{for all } z \text{ in } \mathcal{R}.$$

It follows from the Zero Derivative Theorem that

$$F(z) = c, \quad \text{for all } z \text{ in } \mathcal{R},$$

where c is a complex constant.

Thus $F_1(z) = F_2(z) + c$, for all z in \mathcal{R} .

Section 4

4.1 (a) The standard parametrization of the circle with centre α and radius r is

$$\gamma(t) = \alpha + re^{it} \quad (t \in [0, 2\pi]).$$

Then $\gamma'(t) = rie^{it}$, so that

$$|\gamma'(t)| = r.$$

Thus the required length is

$$\int_0^{2\pi} r dt = 2\pi r.$$

(b) Since $\gamma(t) = t + i \cosh t$ ($t \in [0, 1]$), we have

$$\gamma'(t) = 1 + i \sinh t,$$

and

$$|\gamma'(t)| = \sqrt{1 + \sinh^2 t} = \cosh t \quad (\text{since } \cosh t > 0).$$

Thus the required length is

$$\int_0^1 \cosh t dt = [\sinh t]_0^1 = \sinh 1.$$

4.2 (a) We have $L = 10\pi$, as Γ is a circle of radius 5. To find a value for M note that

$$|e^{3z}| = e^{3 \operatorname{Re} z} \leq e^{15}, \quad \text{for } z \in \Gamma \quad (\text{since } \operatorname{Re} z \leq 5, \text{ for } |z| = 5),$$

and

$$|z^2| = 25, \quad \text{for } z \in \Gamma.$$

Thus

$$\left| \frac{e^{3z}}{z^2} \right| = \frac{|e^{3z}|}{|z^2|} \leq \frac{e^{15}}{25}, \quad \text{for } z \in \Gamma,$$

so we take $M = e^{15}/25$. Since $f(z) = e^{3z}/z^2$ is continuous on $\mathbb{C} - \{0\}$ (by the Quotient Rule) and hence on Γ , it follows from the Estimation Theorem that

$$\left| \int_{\Gamma} \frac{e^{3z}}{z^2} dz \right| \leq \frac{e^{15}}{25} \times 10\pi = \frac{2}{5}\pi e^{15}.$$

(b) We have $L = 40$, as Γ is a square of side 10. To find a value for M note that

$$|e^{3z}| = e^{3 \operatorname{Re} z} \leq e^{15}, \quad \text{for } z \in \Gamma \quad (\text{since } \operatorname{Re} z \leq 5, \text{ for } z \in \Gamma),$$

and

$$|z^2| \geq 25, \quad \text{for } z \in \Gamma.$$

Thus

$$\left| \frac{e^{3z}}{z^2} \right| = \frac{|e^{3z}|}{|z^2|} \leq \frac{e^{15}}{25}, \quad \text{for } z \in \Gamma,$$

so we take $M = e^{15}/25$. Since $f(z) = e^{3z}/z^2$ is continuous on Γ (see part (a)), it follows from the Estimation Theorem that

$$\left| \int_{\Gamma} \frac{e^{3z}}{z^2} dz \right| \leq \frac{e^{15}}{25} \times 40 = \frac{8}{5}e^{15}.$$

4.3 We have $L = 6\pi$, as Γ is a circle of radius 3. To find a value for M note that, by the Triangle Inequality (usual form), we have

$$|3z - 4| \leq |3z| + |-4| = 9 + 4 = 13, \quad \text{for } z \in \Gamma,$$

and, by the Triangle Inequality (backwards form), we have

$$|2z - 5| \geq |2z| - 5 = 6 - 5 = 1, \quad \text{for } z \in \Gamma.$$

Thus

$$\left| \frac{3z - 4}{2z - 5} \right| \leq \frac{13}{1} = 13, \quad \text{for } z \in \Gamma,$$

so we take $M = 13$. Since $f(z) = (3z - 4)/(2z - 5)$ is continuous on $\mathbb{C} - \{5/2\}$ and hence on Γ , it follows from the Estimation Theorem that

$$\left| \int_{\Gamma} \frac{3z - 4}{2z - 5} dz \right| \leq 13 \times 6\pi = 78\pi.$$

4.4 We have $L = 4\pi$, as Γ is a semi-circle of radius 4. To find a value for M note that, by the Triangle Inequality (backwards form), we have

$$|z^2 - 9| \geq |z^2| - 9 = 16 - 9 = 7, \quad \text{for } z \in \Gamma.$$

For the exponential term, we can write

$$\begin{aligned} |e^{2iz}| &= |e^{2i(x+iy)}| \\ &= |e^{2ix}| |e^{-2y}| \\ &= |e^{-2y}| \\ &= e^{-2y} \\ &\leq e^0 = 1, \quad \text{for } z \in \Gamma \quad (\text{since } y \geq 0, \text{ for } z \in \Gamma). \end{aligned}$$

Thus

$$\left| \frac{e^{2iz}}{z^2 - 9} \right| \leq \frac{1}{7}, \quad \text{for } z \in \Gamma,$$

so we take $M = 1/7$. Since $f(z) = e^{2iz}/(z^2 - 9)$ is continuous on $\mathbb{C} - \{3, -3\}$ (by the Composition Rule and Quotient Rule) and hence on Γ , it follows from the Estimation Theorem that

$$\left| \int_{\Gamma} \frac{e^{2iz}}{z^2 - 9} dz \right| \leq \frac{1}{7} \times 4\pi = \frac{4}{7}\pi.$$

SOLUTIONS TO THE EXERCISES

Section 1

There are no exercises for this section.

Section 2

2.1 (a) The standard parametrization for Γ , the line segment from 1 to i , is

$$\gamma(t) = 1 - t + it \quad (t \in [0, 1]);$$

hence

$$\gamma'(t) = i - 1.$$

(i) Here $f(z) = z$, and

$$\begin{aligned} \int_{\Gamma} z \, dz &= \int_0^1 (1 - t + it) \cdot (i - 1) \, dt \\ &= \int_0^1 (-1 + (1 - 2t)i) \, dt \\ &= \int_0^1 (-1) \, dt + i \int_0^1 (1 - 2t) \, dt \\ &= [-t]_0^1 + i [t - t^2]_0^1 \\ &= -1. \end{aligned}$$

(ii) Here $f(z) = \operatorname{Im} z$, and

$$\begin{aligned} \int_{\Gamma} \operatorname{Im} z \, dz &= \int_0^1 (\operatorname{Im} (1 - t + it)) \cdot (i - 1) \, dt \\ &= \int_0^1 t(i - 1) \, dt \\ &= \int_0^1 (-t) \, dt + i \int_0^1 t \, dt \\ &= [-\frac{1}{2}t^2]_0^1 + i [\frac{1}{2}t^2]_0^1 \\ &= \frac{1}{2}(-1 + i). \end{aligned}$$

(Note that this integral is different from

$\operatorname{Im} \left(\int_{\Gamma} f(z) \, dz \right)$, which from part (i) is 0.)

(iii) Here $f(z) = \bar{z}$, and

$$\begin{aligned} \int_{\Gamma} \bar{z} \, dz &= \int_0^1 \overline{(1 - t + it)} \cdot (i - 1) \, dt \\ &= \int_0^1 (1 - t - it) \cdot (i - 1) \, dt \\ &= \int_0^1 (-1 + 2t + i) \, dt \\ &= \int_0^1 (-1 + 2t) \, dt + i \int_0^1 1 \, dt \\ &= [-t + t^2]_0^1 + i [t]_0^1 \\ &= i. \end{aligned}$$

(Again, note that this is different from $\overline{\int_{\Gamma} z \, dz}$.)

Remark Of course, we could have written the above solution using the 'z, dz approach', as we do in part (b).

(b) The standard parametrization for Γ , the unit circle $\{z : |z| = 1\}$, is

$$\gamma(t) = e^{it} \quad (t \in [0, 2\pi]);$$

hence

$$z = e^{it}, \quad dz = ie^{it} \, dt.$$

(i) Here $f(z) = \bar{z} = e^{-it}$, and

$$\begin{aligned} \int_{\Gamma} \bar{z} \, dz &= \int_0^{2\pi} (e^{-it}) \cdot (ie^{it}) \, dt \\ &= i \int_0^{2\pi} 1 \, dt \\ &= i [t]_0^{2\pi} \\ &= 2\pi i. \end{aligned}$$

(ii) Here $f(z) = z^2 = e^{2it}$, and

$$\begin{aligned} \int_{\Gamma} z^2 \, dz &= \int_0^{2\pi} (e^{2it}) \cdot (ie^{it}) \, dt \\ &= \int_0^{2\pi} ie^{3it} \, dt \\ &= \int_0^{2\pi} i(\cos 3t + i \sin 3t) \, dt \\ &= \int_0^{2\pi} (-\sin 3t) \, dt + i \int_0^{2\pi} \cos 3t \, dt \\ &= \left[\frac{1}{3} \cos 3t \right]_0^{2\pi} + i \left[\frac{1}{3} \sin 3t \right]_0^{2\pi} \\ &= 0. \end{aligned}$$

(c) The standard parametrization for Γ , the upper half of the circle with centre 0 and radius 2 from 2 to -2 , is

$$\gamma(t) = 2e^{it} \quad (t \in [0, \pi]);$$

hence

$$\gamma'(t) = 2ie^{it}.$$

(i) Here $f(z) = 1/z$, and

$$\begin{aligned} \int_{\Gamma} \frac{1}{z} \, dz &= \int_0^{\pi} \left(\frac{1}{2e^{it}} \right) \cdot (2ie^{it}) \, dt \\ &= i \int_0^{\pi} 1 \, dt \\ &= i [t]_0^{\pi} \\ &= \pi i. \end{aligned}$$

(ii) Here $f(z) = |z|$, and

$$\begin{aligned} \int_{\Gamma} |z| \, dz &= \int_0^{\pi} |2e^{it}| \cdot 2ie^{it} \, dt \\ &= \int_0^{\pi} 4i(\cos t + i \sin t) \, dt \\ &= \int_0^{\pi} (-4 \sin t) \, dt + i \int_0^{\pi} 4 \cos t \, dt \\ &= [4 \cos t]_0^{\pi} + i [4 \sin t]_0^{\pi} \\ &= -8. \end{aligned}$$

2.2 (a) Γ_1 and Γ_2 are not equivalent (the sets Γ_1 and Γ_2 are not equal).

(b) Γ_1 and Γ_2 are equivalent ($\gamma_1(t) = \gamma_2(h(t))$, where $h(t) = \frac{1}{2}t$ ($t \in [0, \pi]$)).

2.3 (a) $\Gamma = \Gamma_1 + \Gamma_2$, where Γ_1 is the line segment from 0 to i and Γ_2 is the line segment from i to $1+i$. We choose to use the standard parametrizations

$$\begin{aligned}\gamma_1(t) &= it \quad (t \in [0, 1]), \\ \gamma_2(t) &= t + i \quad (t \in [0, 1]).\end{aligned}$$

Then $\gamma_1'(t) = i$, $\gamma_2'(t) = 1$. Hence

$$\begin{aligned}\int_{\Gamma} \operatorname{Re} z \, dz &= \int_{\Gamma_1} \operatorname{Re} z \, dz + \int_{\Gamma_2} \operatorname{Re} z \, dz \\ &= \int_0^1 \operatorname{Re}(it) \cdot i \, dt + \int_0^1 \operatorname{Re}(t+i) \cdot 1 \, dt \\ &= \int_0^1 0 \, dt + \int_0^1 t \, dt \\ &= \left[\frac{1}{2} t^2 \right]_0^1 = \frac{1}{2}.\end{aligned}$$

(b) $\Gamma = \Gamma_1 + \Gamma_2$, where Γ_1 is the line segment from 0 to 1 and Γ_2 is the line segment from 1 to $1+i$. We choose to use the standard parametrizations

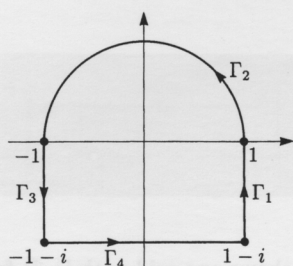
$$\begin{aligned}\gamma_1(t) &= t \quad (t \in [0, 1]), \\ \gamma_2(t) &= 1 + it \quad (t \in [0, 1]).\end{aligned}$$

Then $\gamma_1'(t) = 1$, $\gamma_2'(t) = i$. Hence

$$\begin{aligned}\int_{\Gamma} \operatorname{Re} z \, dz &= \int_{\Gamma_1} \operatorname{Re} z \, dz + \int_{\Gamma_2} \operatorname{Re} z \, dz \\ &= \int_0^1 \operatorname{Re}(t) \cdot 1 \, dt + \int_0^1 \operatorname{Re}(1+it) \cdot i \, dt \\ &= \int_0^1 t \, dt + i \int_0^1 1 \, dt \\ &= \left[\frac{1}{2} t^2 \right]_0^1 + i \left[t \right]_0^1 = \frac{1}{2} + i.\end{aligned}$$

(Note that the integrals in parts (a) and (b) have different values.)

2.4 (a) (i) $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$, where Γ_1 is the line segment from $1-i$ to 1, Γ_2 is the upper half of the unit circle $\{z : |z| = 1\}$, Γ_3 is the line segment from -1 to $-1-i$, and Γ_4 is the line segment from $-1-i$ to $1-i$. (See the figure.)



The standard parametrizations for the paths Γ_1 , Γ_2 , Γ_3 and Γ_4 are

$$\begin{aligned}\gamma_1(t) &= 1 + (t-1)i \quad (t \in [0, 1]), \\ \gamma_2(t) &= e^{it} \quad (t \in [0, \pi]), \\ \gamma_3(t) &= -1 - it \quad (t \in [0, 1]), \\ \gamma_4(t) &= -1 + 2t - i \quad (t \in [0, 1]).\end{aligned}$$

(ii) The reverse contour $\tilde{\Gamma}$ of Γ is

$$\tilde{\Gamma} = \tilde{\Gamma}_4 + \tilde{\Gamma}_3 + \tilde{\Gamma}_2 + \tilde{\Gamma}_1$$

and smooth parametrizations for $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$, $\tilde{\Gamma}_3$ and $\tilde{\Gamma}_4$ are

$$\begin{aligned}\tilde{\gamma}_1(t) &= 1 - ti \quad (t \in [0, 1]), \\ \tilde{\gamma}_2(t) &= e^{i(\pi-t)} \quad (t \in [0, \pi]), \\ \tilde{\gamma}_3(t) &= -1 - i(1-t) \quad (t \in [0, 1]), \\ \tilde{\gamma}_4(t) &= 1 - 2t - i \quad (t \in [0, 1]).\end{aligned}$$

(b) (i) The derivatives of the above parametrizations for Γ_1 , Γ_2 , Γ_3 , Γ_4 are

$$\gamma_1'(t) = i, \quad \gamma_2'(t) = ie^{it}, \quad \gamma_3'(t) = -i, \quad \gamma_4'(t) = 2.$$

Hence

$$\begin{aligned}\int_{\tilde{\Gamma}} \bar{z} \, dz &= \int_{\tilde{\Gamma}_1} \bar{z} \, dz + \int_{\tilde{\Gamma}_2} \bar{z} \, dz + \int_{\tilde{\Gamma}_3} \bar{z} \, dz + \int_{\tilde{\Gamma}_4} \bar{z} \, dz \\ &= \int_0^1 \overline{(1-ti)} \cdot (-i) \, dt + \int_0^{\pi} (\overline{e^{it}}) \cdot (ie^{it}) \, dt \\ &\quad + \int_0^1 \overline{(-1-i(1-t))} \cdot (-i) \, dt + \int_0^1 \overline{(1-2t-i)} \cdot 2 \, dt \\ &= \int_0^1 (1+t-i) \, dt + \int_0^{\pi} i \, dt + \int_0^1 (t+i) \, dt \\ &\quad + \int_0^1 (-2+4t+2i) \, dt \\ &= i \int_0^1 1 \, dt + \int_0^1 (t-1) \, dt + i \int_0^{\pi} 1 \, dt \\ &\quad + \int_0^1 t \, dt + i \int_0^1 1 \, dt \\ &\quad + \int_0^1 (-2+4t) \, dt + i \int_0^1 2 \, dt \\ &= i \left[t \right]_0^1 + \left[\frac{1}{2} t^2 - t \right]_0^1 + i \left[t \right]_0^{\pi} + \left[\frac{1}{2} t^2 \right]_0^1 + i \left[t \right]_0^1 \\ &\quad + \left[-2t + 2t^2 \right]_0^1 + i \left[2t \right]_0^1 \\ &= i - \frac{1}{2} + i\pi + \frac{1}{2} + i + 0 + 2i = (\pi + 4)i.\end{aligned}$$

(ii) Since $f(z) = \bar{z}$ is a continuous function, it follows from the Reverse Contour Theorem that

$$\begin{aligned}\int_{\tilde{\Gamma}} \bar{z} \, dz &= - \int_{\Gamma} \bar{z} \, dz \\ &= -(\pi + 4)i.\end{aligned}$$

Section 3

3.1 The Fundamental Theorem of Calculus may be used to evaluate $\int_{\Gamma} f(z) dz$, where Γ is any contour from $-i$ to i , for the functions f in parts (a)–(h). In each case f is continuous on \mathbb{C} and has a primitive on \mathbb{C} .

(a) $\int_{\Gamma} 1 dz = [z]_{-i}^i = i - (-i) = 2i.$

(b) $\int_{\Gamma} z dz = [\frac{1}{2}z^2]_{-i}^i = \frac{1}{2}i^2 - \frac{1}{2}(-i)^2 = 0.$

(c) $\int_{\Gamma} (5z^4 + 3iz^2) dz = [z^5 + iz^3]_{-i}^i = (i + 1) - (-i - 1) = 2 + 2i.$

(d) $\int_{\Gamma} (1 + 2iz)^9 dz = [(1 + 2iz)^{10} / (10 \times 2i)]_{-i}^i = ((-1)^{10} - (3)^{10}) / (20i) = \frac{3^{10} - 1}{20}i.$

(e) $\int_{\Gamma} e^{-iz} dz = [e^{-iz} / (-i)]_{-i}^i = (e - e^{-1}) / (-i) = 2i \sinh 1.$

(f) $\int_{\Gamma} \sin z dz = [-\cos z]_{-i}^i = -\cos i + \cos(-i) = 0.$

(g) We have

$$f(z) = ze^{z^2} = z \exp(z^2) = \frac{1}{2}(2z) \exp(z^2).$$

Hence a primitive of f is

$$F(z) = \frac{1}{2} \exp(z^2),$$

and

$$\int_{\Gamma} ze^{z^2} dz = [\frac{1}{2} \exp(z^2)]_{-i}^i = \frac{1}{2}(e^{-1} - e^{-1}) = 0.$$

(h) We have

$$f(z) = z^3 \cosh(z^4) = \frac{1}{4}(4z^3) \cosh(z^4).$$

Hence a primitive of f is

$$F(z) = \frac{1}{4} \sinh(z^4),$$

and

$$\int_{\Gamma} z^3 \cosh(z^4) dz = [\frac{1}{4} \sinh(z^4)]_{-i}^i = \frac{1}{4}(\sinh 1 - \sinh 1) = 0.$$

(i) Let $f(z) = z$, $g(z) = e^z$. Then f and g are entire, and f' and g' are entire and hence continuous. Then, by Integration by Parts (Theorem 3.3), we have

$$\begin{aligned} \int_{\Gamma} ze^z dz &= [ze^z]_{-i}^i - \int_{\Gamma} 1 \cdot e^z dz \\ &= (ie^i - (-i)e^{-i}) - \int_{\Gamma} e^z dz \\ &= i(e^i + e^{-i}) - [e^z]_{-i}^i \\ &= 2i \cos 1 - (e^i - e^{-i}) \\ &= 2i \cos 1 - 2i \sin 1 \\ &= 2(\cos 1 - \sin 1)i. \end{aligned}$$

3.2 (a) Let $f(z) = 1/z$, $F(z) = \text{Log } z$ and $\mathcal{R} = \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$. Then f is continuous on \mathcal{R} , F is a primitive of f on \mathcal{R} , and Γ is a contour in \mathcal{R} . Thus, by the Fundamental Theorem of Calculus,

$$\begin{aligned} \int_{\Gamma} \frac{1}{z} dz &= [\text{Log } z]_{-i}^i \\ &= \text{Log } i - \text{Log}(-i) \\ &= \frac{1}{2}\pi i - (-\frac{1}{2}\pi i) = \pi i. \end{aligned}$$

(b) Let $f(z) = \sqrt{z}$, $F(z) = \frac{2}{3}z^{3/2}$ and $\mathcal{R} = \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$. Then f is continuous on \mathcal{R} , F is a primitive of f on \mathcal{R} , and Γ is a contour in \mathcal{R} . Thus, by the Fundamental Theorem of Calculus,

$$\begin{aligned} \int_{\Gamma} \sqrt{z} dz &= [\frac{2}{3}z^{3/2}]_{-i}^i \\ &= \frac{2}{3}(i^{3/2} - (-i)^{3/2}) \\ &= \frac{2}{3}(\exp(\frac{3}{2}\text{Log } i) - \exp(\frac{3}{2}\text{Log}(-i))) \\ &= \frac{2}{3}(\exp(\frac{3}{4}\pi i) - \exp(-\frac{3}{4}\pi i)) \\ &= \frac{2}{3}(2i \sin \frac{3}{4}\pi) \\ &= \frac{2\sqrt{2}}{3}i. \end{aligned}$$

(c) The function $f(z) = \sin^2 z$ is continuous and has an entire primitive $F(z) = \frac{1}{2}(z - \frac{1}{2}\sin 2z)$. (This primitive is found by writing $\sin^2 z = \frac{1}{2}(1 - \cos 2z)$.) Thus, by the Closed Contour Theorem,

$$\int_{\Gamma} \sin^2 z dz = 0.$$

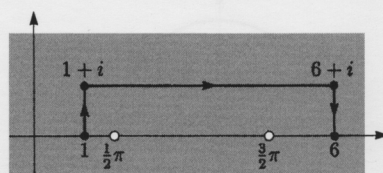
(d) Let $f(z) = 1/z^3$, $F(z) = -1/(2z^2)$ and $\mathcal{R} = \mathbb{C} - \{0\}$. Then f is continuous on \mathcal{R} , F is a primitive of f on \mathcal{R} , and Γ is in \mathcal{R} . Thus, by the Closed Contour Theorem,

$$\int_{\Gamma} \frac{1}{z^3} dz = 0.$$

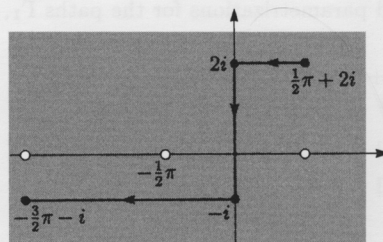
3.3 The domain of \tan is the region

$$\mathcal{R} = \mathbb{C} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}.$$

(a) The figure shows one grid path in \mathcal{R} from 1 to 6.



(b) The figure shows one grid path in \mathcal{R} from $\frac{1}{2}\pi + 2i$ to $-\frac{3}{2}\pi - i$.



Section 4

4.1 In each case we check that the hypotheses of the Estimation Theorem apply; these are

f is continuous on Γ (of length L), and

$$|f(z)| \leq M, \quad \text{for } z \in \Gamma.$$

In each case Γ is the circle $\{z : |z| = 3\}$, so $L = 6\pi$.

(a) The function $f(z) = z + 2$ is continuous on \mathbb{C} and hence on Γ .

To find a value for M note that, by the Triangle Inequality (usual form), we have

$$\begin{aligned} |z + 2| &\leq |z| + |2| \\ &= 3 + 2 = 5, \quad \text{for } z \in \Gamma, \end{aligned}$$

so we take $M = 5$. It follows from the Estimation Theorem that

$$\left| \int_{\Gamma} (z + 2) dz \right| \leq 5 \times 6\pi = 30\pi.$$

(b) The function $f(z) = 1/(z + 2)$ is continuous on $\mathbb{C} - \{-2\}$ and hence on Γ .

To find a value for M note that, by the Triangle Inequality (backwards form), we have

$$\begin{aligned} |z + 2| &\geq ||z| - |2|| \\ &= 3 - 2 = 1, \quad \text{for } z \in \Gamma. \end{aligned}$$

Thus

$$\left| \frac{1}{z + 2} \right| \leq 1, \quad \text{for } z \in \Gamma,$$

so we take $M = 1$. It follows from the Estimation Theorem that

$$\left| \int_{\Gamma} \frac{1}{z + 2} dz \right| \leq 1 \times 6\pi = 6\pi.$$

(c) The function $f(z) = (z - 3)/(z + 2)$ is continuous on $\mathbb{C} - \{-2\}$ and hence on Γ .

To find a value for M note that, by the Triangle Inequality (usual form), we have

$$\begin{aligned} |z - 3| &\leq |z| + |-3| \\ &= 3 + 3 = 6, \quad \text{for } z \in \Gamma, \end{aligned}$$

and, by the Triangle Inequality (backwards form), we have

$$\begin{aligned} |z + 2| &\geq ||z| - |2|| \\ &= 3 - 2 = 1, \quad \text{for } z \in \Gamma. \end{aligned}$$

Thus

$$\left| \frac{z - 3}{z + 2} \right| \leq \frac{6}{1} = 6, \quad \text{for } z \in \Gamma,$$

so we take $M = 6$. It follows from the Estimation Theorem that

$$\left| \int_{\Gamma} \frac{z - 3}{z + 2} dz \right| \leq 6 \times 6\pi = 36\pi.$$

(d) The function $f(z) = (z^2 + 4)/(z^2 - 4)$ is continuous on $\mathbb{C} - \{2, -2\}$ and hence on Γ .

To find a value for M note that, by the Triangle Inequality (usual form), we have

$$\begin{aligned} |z^2 + 4| &\leq |z^2| + |4| \\ &= 9 + 4 = 13, \quad \text{for } z \in \Gamma, \end{aligned}$$

and, by the Triangle Inequality (backwards form), we have

$$\begin{aligned} |z^2 - 4| &\geq ||z^2| - |4|| \\ &= 9 - 4 = 5, \quad \text{for } z \in \Gamma. \end{aligned}$$

Thus

$$\left| \frac{z^2 + 4}{z^2 - 4} \right| \leq \frac{13}{5}, \quad \text{for } z \in \Gamma,$$

so we take $M = \frac{13}{5}$. It follows from the Estimation Theorem that

$$\left| \int_{\Gamma} \frac{z^2 + 4}{z^2 - 4} dz \right| \leq \frac{13}{5} \times 6\pi = \frac{78}{5}\pi.$$

(e) The function $f(z) = e^{z-3}$ is continuous on \mathbb{C} (by the Composition Rule) and hence on Γ .

To find a value for M note that

$$\begin{aligned} |e^{z-3}| &= e^{\operatorname{Re}(z-3)} \\ &= e^{x-3} \quad (z = x + iy) \\ &\leq e^0 = 1, \quad \text{for } z \in \Gamma, \end{aligned}$$

so we take $M = 1$. It follows from the Estimation Theorem that

$$\left| \int_{\Gamma} e^{z-3} dz \right| \leq 1 \times 6\pi = 6\pi.$$

(f) The function $f(z) = \sin z/(1 + z^2)$ is continuous on $\mathbb{C} - \{i, -i\}$ and hence on Γ .

To find a value for M note that, by the Triangle Inequality (usual form), we have

$$\begin{aligned} |\sin z| &= \frac{1}{2} |e^{iz} - e^{-iz}| \\ &\leq \frac{1}{2} (|e^{iz}| + |e^{-iz}|) \\ &= \frac{1}{2} (e^{\operatorname{Re}(iz)} + e^{\operatorname{Re}(-iz)}) \\ &= \frac{1}{2} (e^{-y} + e^y) \quad (z = x + iy) \\ &\leq \frac{1}{2} (e^3 + e^3) = e^3, \quad \text{for } z \in \Gamma; \end{aligned}$$

and, by the Triangle Inequality (backwards form), we have

$$\begin{aligned} |1 + z^2| &\geq |1 - |z^2|| \\ &= |1 - 9| = 8, \quad \text{for } z \in \Gamma. \end{aligned}$$

Thus

$$\left| \frac{\sin z}{1 + z^2} \right| \leq \frac{e^3}{8}, \quad \text{for } z \in \Gamma,$$

so we take $M = \frac{1}{8}e^3$. It follows from the Estimation Theorem that

$$\left| \int_{\Gamma} \frac{\sin z}{1 + z^2} dz \right| \leq \frac{1}{8}e^3 \times 6\pi = \frac{3}{4}e^3\pi.$$